Question 1. We need to show that the quotient space $cX = S^1 \times I / S^1 \times 1$ is homeomorphic to D^2. Here $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Notice that S^1 is the boundary ∂D^2 of D^2. It is also the base $S^1 \times 0$ of the cone cS^1. I will prove something stronger than what's required. Instead of identifying $S^1 \times 0$ with ∂D^2 via the identity map, I will choose any homeomorphism

$$h\colon S^1 \times 0 \longrightarrow \partial D^2$$

and show that there is a homeomorphism

$$H\colon cS^1 \longrightarrow D^2$$

which *extends* h, that is a homeomorphism H which is precisely h when restricted to $S^1 \times 0$. Such constructions are called *relative*.

Proof. Define H by $H(S^1 \times 1) = (0,0)$ and mapping each element $x \times [0,1]$ linearly onto the radial segment from $(0,0)$ to $h(x)$. This means that $H(x,t) = (1-t)h(x)$, which is a vector equation in $\mathbb{R}^n$.

If U is an open subset of D^2 disjoint from $(0,0)$ then $H^{-1}(U) \subset S^1 \times [0,1)$ where H is injective, and so $H^{-1}(U)$ is clearly open in cS^1. If U is a neighborhood of $(0,0)$ then there is a number ε such that the metric ball

$$B((0,0),\varepsilon) = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq \varepsilon\}$$

is contained in U. This shows that $S^1 \times (1-\varepsilon, 1]$, which is the collar of thickness ε on the top lid of $S^1 \times I$, is contained entirely in $H^{-1}(U)$. So $H^{-1}(U)$ is a neighborhood of the cone point $S^1 \times 1 / S^1 \times 1$ in cS^1. This shows that H is continuous. The same kind of argument shows that its inverse is continuous. □

Question 2. Given CS^1 and D^2 as before, define $H(S^1 \times 1 \cup x_0 \times [0,1]) = (0,1)$. Then for any homeomorphism $h\colon S^1 \times 0 \to \partial D^2$ which sends $x_0 \times 0$ to $(0,1)$, define H by mapping each $x \times [0,1]$ linearly onto the straight segment from $h(x)$ to $(0,1)$. This means

$$H(x,t) = (1-t)(0,1) + th(x).$$

This is a homeomorphism extending h.

Question 3. Let

$$H\colon S^1 \times [-1,1]/S^1 \times \{-1\}/S^1 \times \{1\} \longrightarrow S^2$$

be the radial projection perpendicular to the z-axis. So

$$H([\overline{x}, t]) = (\sqrt{1-t^2}\,\overline{x}, t).$$

Of course, when $t = \pm 1$, H sends the lids onto the poles $(0,0,1)$ and $(0,0,-1)$, as it should.