MAT 540A Fall 2014 **Take-home test No. 1**

Please turn in the solutions on Monday, September 15, in class.

Question 1 (2 points)**.** This is one of the homework problems. I would like to see your solutions. Classify all topologies on the set with three elements up to homeomorphisms. This means you need to find out which pairs of topologies have pairs of continuous maps between them that are inverses of each other. (You might find it useful to use the easy fact that being homeomorphic is an equivalence relation. So what you are actually doing is sorting out the equivalence classes of these topologies.)

Question 2 (2 points)**.** I have used the following fact in class a lot but we haven't proved it. It is a characterization of the usual topology on the real line. So now I am asking to show that every open set in $\mathbb{R}$ is the union of *disjoint* open intervals (a, b) where we allow $a = -\infty$ and $b = \infty$. [Just to be sure we agree on definitions: an open subset of $\mathbb{R}$ is a subset U with the property that for each $x \in U$ there is an interval (a, b) so that $x \in (a, b) \subset U$. So the usual topology is simply the topology generated by the basis consisting of open intervals (a, b).]

Question 3 (1 points)**.** Use only the topology facts from this course to show that $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x + y$ is a continuous function.

Question 4 (5 points)**.** For two integers a and b such that $a \neq 0$, define $\mathcal{B}$ to be the collection of subsets $A(a, b) = \{an + b | n \in \mathbb{Z}\}$ of the integers $\mathbb{Z}$. The subsets are called the arithmetic progressions, of course.
(a) Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.
(b) Show that a cofinite set (= the complement of a finite set) cannot be closed in this topology.
(c) By definition, $A(a, b)$ is open in this topology. Show that $A(a, b)$ is also closed.
(d) Show that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} A(p, 0).$$

(e) Explain how it follows that there are infinitely many prime numbers.