Question 1. Before the solution, one comment. A homeomorphism is a bijection between sets that also gives a bijection between topologies. That follows directly from the definitions. One consequence is that the homeomorphism class is determined by the pattern of the nesting between the open sets (really, the specific partial order of inclusions) in the topology. Relabelings of the three elements that give the same pattern give a homeomorphism. The number of such possible relabelings for each pattern give the number of the topologies in this homeomorphism class.

So here we go. The factor in front is the number of permutations of the labels that change the topology while preserving the homeomorphism class.

1. $1 \times \{\emptyset, X\}$ (the trivial topology),
2. $1 \times \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ (the descrete topology),
3. $3 \times \{\emptyset, X, \{a\}\}$,
4. $3 \times \{\emptyset, X, \{a\}, \{b,c\}\}$,
5. $3 \times \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$,
6. $3 \times \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$,
7. $3 \times \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$,
8. $6 \times \{\emptyset, X, \{a\}, \{a,b\}\}$,
9. $6 \times \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$.

This gives 29 topologies on $\{a,b,c\}$, but only 9 up to homeomorphism.

Question 2. Given an open set in $\mathbb{R}$, we know it is the union of a family of open intervals (a_α, b_α). The claim is that the union U of all (a_α, b_α) is in fact organized as the union of disjoint intervals. Suppose x is in the complement $\mathbb{R} \setminus U$ then there is no α such that $x \in (a_\alpha, b_\alpha)$. This means that if $x < b_\alpha$ for some α then automatically $x < a_\alpha$ for the same α. Similarly, if $x > a_\alpha$ then $x > b_\alpha$. This classifies all intervals into those that are to the right of x (1st case) and to the left of x (2nd case). If d_x is the largest number such that $x \le d_x \le a_\alpha$ for some α, and c_x is the smallest number such that $b_\beta \le c_x \le x$ then $d_x = a_\alpha$, and $c_x = b_\beta$ for some β, and $[b_\beta, a_\alpha]$ is in the complement of U with $x \in [b_\beta, a_\alpha]$. This construction gives a union C of closed disjoint intervals $[b_\beta, a_\alpha]$ whose union is $\mathbb{R} \setminus U$ (because x was arbitrary). The complement $\mathbb{R} \setminus C = U$, the union of disjoint open intervals.

Question 3. Our map is $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x + y$. It suffices to check that the preimage of a basic set in $\mathbb{R}$ is open in $\mathbb{R}^2$. What does $f^{-1}(a,b)$ look like? It's an infinite slant strip without the edges, with the core which is the line $y = \frac{a+b}{2} - x$. It is the union of slant open squares centered at points $(x, \frac{a+b}{2} - x)$ and radius $\frac{b-a}{2}$. These are basic open metric balls for $\mathbb{R}^2$ using the taxi-cab distance. We know they are all open, so the preimage $f^{-1}(a,b)$ is open.

Question 4. (a) First let's check that for all $k \in \mathbb{Z}$ there exists some $A(a,b)$ so that $x \in A(a,b)$. For that we just notice that $\mathbb{Z} = A(1,0)$. For the second property, given $x \in A(a,b) \cap A(c,d)$, we want some $A(e,f) \subset A(a,b) \cap A(c,d)$ which contains x. Notice that instead of $x \in A(a,b) \cap A(c,d)$ we may equivalently write $x \in A(a,x) \cap A(c,x)$ so

$x \in A(ac, x) \subset A(a, b) \cap A(c, d)$. This means we can choose $e = ac$, $f = x$ and be done. [If we want to be really efficient then we can realize that $A(a, x) \cap A(c, x) = A(\mathrm{lcm}(a, c), x)$. This gives another choice $e = \mathrm{lcm}(a, c)$, $f = x$.]

(b) Suppose a cofinite set C is closed. Then some finite set $\mathbb{Z} \setminus C$ is open. This means it is a union of basis elements. But every non-empty basis element $A(a, b)$ is infinite, so the first statement is impossible.

(c) The key is that $A(a, b)$ and a finite number of its (disjoint!) translates completely covers $\mathbb{Z}$: first notice that for each index i such that $0 \leq i < |a|$ we have $A(a, b) \cap A(a, b + i) = \emptyset$, then also notice that $\mathbb{Z}$ is the union of all $A(a, b + i)$ for $0 \leq i < |a|$. In particular, $\mathbb{Z}$ is the disjoint union of $A(a, b)$ and the separate union of all $A(a, b + i)$ for $1 \leq i < |a|$. The latter set is open as the union of open sets, so $A(a, b)$ is closed.

(d) Of course, no prime divides 1 or -1, so 1 and -1 are not in any $A(p, 0)$, so neither is in the union. For all other integers n, some prime q divides n, so $n \in A(q, 0)$, so n is in the union in the display. We have shown two opposite inclusions as required.

(e) By (b) a cofinite set cannot be expressed as the finite union of closed sets, but $\mathbb{Z} \setminus \{-1, 1\}$ is cofinite, and $A(p, 0)$ is closed by (c). Therefore, $\mathbb{Z} \setminus \{-1, 1\}$ must be an infinite union of sets of the type $A(p, 0)$, so there must be an infinite number of primes.