Localized operations on T-equivariant oriented cohomology of projective homogeneous varieties

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Moment graphs

[Goresky-Kottwitz-MacPherson, *Invent. Math.*, 1998],
[Braden-MacPherson, *Math. Ann.*, 2001],
[Fiebig, *Adv. Math.*, 2008],
[Fiebig, *J. Amer. Math. Soc.*, 2011]

Let Σ be a root system (also finite real non-crystallographic)

$$\Sigma = \Sigma^+ \amalg \Sigma^-$$

By a moment graph $\mathcal{G}$ with respect to Σ we call a directed labelled graph:

- Vertices are given by elements of some poset $(V, \leq)$
- Directed edges E are labelled by positive roots, i.e., we are given a label function $l\colon E \to \Sigma^+$ (there are no multiple edges)
- The direction of edges respects the partial order, i.e., $v \to w \in E \implies v \leq w,\ v \neq w$ (there are no directed cycles)

Example: Bruhat graphs

Consider the usual Bruhat poset $(W,\leq)$, where W is the Weyl (Coxeter) group of Σ.

The data

$$(V,\leq) := (W,\leq),$$

$$E := \{w \to s_\alpha w \mid w \leq s_\alpha w,\ w \in W,\ \alpha \in \Sigma^+\}$$

$$\text{and} \quad l(w \to s_\alpha w) := \alpha$$

define a moment graph.
Observe that the transitive closure of E gives the Bruhat order '$\leq$' on W and E contains all cover relations of the Bruhat order.

One can also look at the parabolic case $(W^P,\leq)$ and... even at the double parabolic case $(W_Q \setminus W/W_P,\leq)$.

Formal group laws

R a commutative (graded) unital ring.

$$x +_F y = F(x,y) \in R[[x,y]]$$

is a (one-dimensional) commutative formal group law over R if

$$(x +_F y) +_F z = x +_F (y +_F z), \quad x +_F y = y +_F x, \quad x +_F 0 = x.$$

Universal formal group law

$$F_U(x,y) = x + y + \sum_{i,j\geq 1} a_{ij} x^i y^j, \qquad \deg a_{ij} = 1 - i - j$$

over the Lazard ring $\mathbb{L} = \mathbb{Z}\langle a_{ij}\rangle/(\text{relations of f.g.l.})$. So to give F/R is equivalent to give a (evaluation) map $\mathbb{L} \to R$.

- $F(x,y) = x + y$ (additive) over $R = \mathbb{Z}$
- $F(x,y) = x + y - \beta xy$ (multiplicative) over $\mathbb{Z}[\beta]$, $\deg \beta = -1$.

Generalized structure algebra

[Fiebig, *Adv. Math.*, 2008],

[Devyatov, Lanini, Z. *Documenta*, 2019]

F a formal group law over R

Λ any intermediate lattice between roots and weights of Σ

- Σ is crystallographic and F is any, or
- Σ is any (finite real) and F is additive
  (here Λ is a free module of finite rank over the coefficient ring of Σ)

$S := S_F(\Lambda) = R[[x_\lambda]]_{\lambda\in\Lambda}/(x_0, x_{\mu+\lambda} = x_\mu +_F x_\lambda)$ the formal group ring

$\mathcal{G} := ((V, \leq), l\colon E \to \Sigma^+)$ a moment graph

The submodule of the free S-module $\bigoplus_{x\in V} S$

$$\mathcal{Z}(\mathcal{G}, F) := \left\{ \sum_v z_v f_v \;\text{ s.t. }\; \begin{matrix} z_{l(v\to w)} \mid z_v - z_w \\ \forall v \to w \in E \end{matrix} \right\}$$

together with the coordinate-wise multiplication is called the (generalized) structure algebra of $\mathcal{G}$ and F.

Equivariant generalized (oriented) theories

[Totaro, 1999]

[Heller-Malagon-Lopez, J. Reine Angew. Math. 2013], [Krishna,..]

[Karpenko, Mekurjev, 2020]

Let G be a split simple (not necessarily simply-connected) linear algebraic group over k, $char(k) = 0$, e.g.,

$$SL_n, PGL_n, Spin_n, HSpin_{4n}, \dots$$

Consider smooth G-varieties over k with G-equivariant maps as morphisms.

Let $h_G(-)$ be a G-equivariant algebraic oriented Borel-Moore homology theory obtained from $h(-)$ via the Borel construction, e.g.,

$$CH_G(-), K_G(-), \Omega_G(-), CK_G(-), \dots$$

Quillen's formula

A key property of $h(-)$ and, hence, of $h_G(-)$ is that

It has characteristic classes which satisfy the Quillen formula for the tensor product of line bundles. Namely,

$$c_1^h(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1^h(\mathcal{L}_1) +_F c_1^h(\mathcal{L}_2),$$

where

- c_1^h is the first characteristic class in the theory $h(X)$,
- $\mathcal{L}_i$ is a line bundle over X and
- $F(x, y) \in R[[x, y]]$ is the associated formal group law over $R = h(pt)$.

Correspondence

Given F over R define $h(-) := \Omega(-) \otimes_{\mathbb{L}} R$.
So there is

Formal group laws $\longleftrightarrow$ Free oriented cohomology theories

Universal formal group law $\longleftrightarrow$ algebraic cobordism Ω

$F(x,y) = x + y - \beta xy$ $\longleftrightarrow$ Connective K-theory CK

In particular,
$\beta = 0$ corresponds to the Chow theory CH (usual singular cohomology)
$\beta = 1$ corresponds to the usual K-theory

Geometric case

[Kostant, Kumar, 1986,1990]

[Bressler, Evans] ...

[Ganter, Ram] ...

[Calmès, Z., Zhong,..2019]

- Let G be a split reductive group over k. Let T be a split maximal torus in G and let T^* be its group of characters.
- Let h_T be a T-equivariant oriented cohomology theory corresponding to the formal group law F over $R = h(pt)$.
- Let $\mathcal{G}$ be the Bruhat graph for the root system of (G, B) and $\mathcal{Z}(\mathcal{G}, F)$ be the structure algebra over $S = R[[T^*]]_F$.

Then

$$h_T(pt)^\wedge \simeq S_F(\Lambda) \text{ and } h_T(G/B)^\wedge \simeq \mathcal{Z}(\mathcal{G}, F).$$

- $CH_T(pt) \simeq Sym(T^*)$; $CH_T(G/B)^\wedge \simeq \mathcal{Z}(\mathcal{G}, x+y)$
- $K_T(pt) \simeq \mathbb{Z}[T^*]$; $K_T(G/B)^\wedge \simeq \mathcal{Z}(\mathcal{G}, x+y-xy)$
- $CK_T(G/B)^\wedge \simeq \mathcal{Z}(\mathcal{G}, x+y-\beta xy)$

Riemann-Roch formalism
[SGA6],..., [Panin, 2000]

Morphisms of generalized (oriented) cohomology theories are natural transformations

$$\mathcal{F}\colon h_1(-) \to h_2(-)$$

that preserve orientations $\iff$ characteristic (Euler) classes $\iff$ push-forward structures (perfect integrations).

Riemann-Roch for $\mathcal{F}\colon h_1 \to h_2 \iff$ behaviour of $\mathcal{F}$ with respect to push-forwards

Example: $h_1(-) = K(-)$ and $h_2(-) = CH(-, \mathbb{Q})$, $\mathcal{F} = ch$ is the Chern character.

Given an oriented cohomology theory $h(-)$ denote by $\mathcal{E}nd(h)$ the ring of endomorphisms (natural transformations) preserving the orientation and call it the ring of operations on $h(-)$.

- $\mathcal{E}nd(\Omega) = \langle$ Landweber-Novikov operations $\rangle$,
- $\mathcal{E}nd(CK) = \langle$ Adams operations $\rangle$,
- $\mathcal{E}nd(Ch) = \langle$ Chow traces of L.N. operations$\rangle$,
  where $Ch(-) := CH(-;\mathbb{Z}/p\mathbb{Z})$ for a fixed prime p.
  (Reduced power operations and Steenrod operations appear as examples of the Chow traces.)

Goals

I. Describe these operations on $h(G/P)$ in combinatorial terms, e.g., acting on Schubert basis

Steenrod operations: [Duan-Zhao, Compositio, 2007]
Landweber-Novikov operations: [Calmès-Petrov-Z., Ann.Sci.Ec.Norm., 2013]

II. Extend these operations to the T-equivariant context, i.e., to $h_T(G/P)$ or, more generally, to $\mathcal{Z}(\mathcal{G}, F)$.

Shown in [Garibaldi-Petrov-Semenov, Duke, 2016] that the existence/use of such localized operations for Chow theory allows to compute the usual Reduced power/Steenrod operations more efficiently.

Extension to structure algebras

[Edidin, Graham, Duke, 2000]

[Anderson, Gonzales, Payne, 2019]

Using the functoriality of the formal group ring (with respect to morphisms of formal group laws) we first construct the operation $\mathfrak{C}^F_{pt}$ on the formal group ring.
We then take the direct sum $\mathfrak{C}^F = \oplus_v \mathfrak{C}^F_v$ of operations over all vertices $v \in V$ of the moment graph.

The key point is to prove that it respects the relations of the structure algebra:

Theorem.[Z., 2020] The direct sum $\mathfrak{C}^F$ restricts to

$$\mathfrak{C}^F : \mathcal{Z}(\mathcal{G}, F) \to \mathcal{Z}(\mathcal{G}, F).$$

Such operations are called localized operations.

Properties

- The localized operations commute with the equivariant characteristic map $c_F\colon S \to \mathcal{Z}(\mathcal{G}, F)$, i.e.

$$c_F \circ \mathfrak{C}^F_{pt} = \mathfrak{C}^F \circ c_F.$$

- Under the forgetful map $h_T(-) \to h(-)$ the localized operations restrict to the usual operations.

$Q := S[\frac{1}{char.classes}]$, $Q_W := Q \otimes R[W]$, where W is the Weyl group
Define the Hecke action of Q_W on the dual Q_W^* as follows:

$$(z \bullet f)(z') := f(z'z), \quad z, z' \in Q_W,\ f \in Q_W^*.$$

This action restricts to the action of the subalgebra of push-pull elements on the structure algebra $\mathcal{Z} = \mathcal{Z}(\mathcal{G}, F) = h_T(G/B)$.

The push-pull element Y_α acts on $\mathcal{Z}$ as follows

$$Y_\alpha \bullet (\sum_v z_v f_v) = \sum_v (\kappa_{v(\alpha)} z_v + \Delta_{v(\alpha)}(z_v)) f_v,$$

$\kappa_{v(\alpha)} = \frac{1}{x_{-v(\alpha)}} + \frac{1}{x_{v(\alpha)}} \in S$ and $\Delta_{v(\alpha)}(z_v) = (z_{s_{v(\alpha)}v} - z_v)/x_{v(\alpha)} \in S$.

Riemann-Roch type formula for the Hecke action

We say that F is of additive type if its formal inverse coincides with the usual additive inverse, i.e. $-_F x = -x$.

Suppose $\phi\colon F_1/R_1 \to F_2/R_2$ is a morphism of FGLs of additive type. Let $\mathfrak{C}\colon (h_1)_T(G/B) \to (h_2)_T(G/B)$ be the localized operation induced by ϕ.

Theorem [Z.]

$$\mathrm{itd}_\phi(x_\alpha) \bullet \mathfrak{C}(Y_\alpha \bullet z) = Y_\alpha \bullet \mathfrak{C}(z), \quad \text{for all } z \in (h_1)_T(G/B).$$

Algorithm to compute operations

The action of $\mathfrak{C}$ on the Schubert basis $\{\zeta_{I_w}\}_w$ of $(h_1)_T(G/B)$ can be computed as follows:

- First one computes the matrix M expressing ζ_{I_w} as $\sum_v p_v^{I_w} f_v$ of $(h_1)_T(G/B)$. For this one uses the inductive formulas involving push-pull operators.
- Then one computes $p_v = \mathfrak{C}(p_v^{I_w})$ in S_2 for each of the coordinates and obtains the element $x = \sum_v p_v f_v$ of $(h_2)_T(G/B)$.
- Finally, one finds the matrix M^{-1} and computes $(p_v)_v \cdot M^{-1}$, hence, expressing x in terms of the Schubert basis $\{\zeta_{I_w}\}_w$ in $(h_2)_T(G/B)$.

Adams operations

Following [Vishik, 2017] given $h(-)$, we define the integral Adams operations

$$\Psi_k\colon h(-) \to h(-), \quad k \in \mathbb{Z}$$

as the multiplicative operations corresponding to $k \cdot_F -$.

Observe that for the K_0 it gives the usual Adams operations and for the connective K-theory CK it gives the Adams operations studied in [Merkurjev-Vishik, 2020].

On $S = CK_T(pt)$ over $R = \mathbb{Z}[\beta]$ it is defined by

$$\beta\Psi_k(x_\lambda) = (1 - (1 - \beta x_\lambda)^k), \quad \beta x_\lambda = (1 - e^\lambda),\ \lambda \in \Lambda.$$

Example of computation for CK

Let $G = PGL_3$ be the adjoint simple group of type A_2. Then the root lattice $\Lambda = T^*$ has a basis $\Pi = \{\alpha_1, \alpha_2\}$ consisting of simple roots. We have $W = \langle s_1, s_2 \rangle$ with $s_1(\alpha_2) = \alpha_1 + \alpha_2$ and $\dim G/B = 3$.

Suppose $F(x, y) = x + y - \beta xy$ is a multiplicative formal group law over $\mathbb{Z}[\beta]$ which corresponds to the connective K-theory. We then have $S = \mathbb{Z}[\beta][[\Lambda]]_F^\wedge$ and $S[\beta^{-1}] \simeq \mathbb{Z}[\beta^{\pm 1}][\Lambda]^\wedge$ is the group ring of Λ, where $\beta x_\lambda \leftrightarrow 1 - e^{-\lambda}$. Observe that $x_\lambda = c_1^{CK}(\mathcal{L}(\lambda))$ of the respective line bundle.

Consider the class $x_\Pi = x_{-\alpha_1} x_{-\alpha_1-\alpha_2} x_{-\alpha_2}$ so that $\beta^3 x_\Pi = (1 - e^{\alpha_1})(1 - e^{\alpha_1+\alpha_2})(1 - e^{\alpha_2})$.

Applying the operators Y_{α_i}'s to $[pt] = x_\Pi f_1$ we obtain the Schubert basis $\{\zeta_w\}_w$. In particular,

$$\zeta_{s_1 s_2} = Y_{\alpha_2} \bullet (Y_{\alpha_1} \bullet [pt]) = Y_{\alpha_2} \bullet (\tfrac{x_\Pi}{x_{-\alpha_1}}(f_1 + f_{s_1}))$$
$$= x_{-\alpha_1-\alpha_2}(f_1 + f_{s_2}) + x_{-\alpha_2}(f_{s_1} + f_{s_1 s_2}) \in CK^1_T(G/B).$$

We apply the Adams operation Ψ_2 to the coefficients:

$$\Psi_2(x_{-\alpha_1-\alpha_2}) = 2x_{-\alpha_1-\alpha_2} - \beta x^2_{-\alpha_1-\alpha_2} \text{ and } \Psi_2(x_{-\alpha_2}) = 2x_{-\alpha_2} - \beta x^2_{-\alpha_2}.$$

Hence,

$$\Psi_2(\zeta_{s_1 s_2}) = (2 - \beta x_{-\alpha_2})\zeta_{s_1 s_2} + \beta(\beta x_{-\alpha_2} - 1)\zeta_{s_2}$$
$$= (1 + e^{\alpha_2})\zeta_{s_1 s_2} - \beta e^{\alpha_2}\zeta_{s_2}.$$

After applying the augmentation map $(e^\lambda \mapsto 1)$ we obtain

$$\Psi_2(\zeta_{s_1 s_2}) = 2\zeta_{s_1 s_2} - \beta\zeta_{s_2} \in CK^1(G/B).$$

Thank You!