Whittaker functions from motivic Chern classes

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Plan:

1) Motivic Chern class in K-theory

2) Iwahori-Whittaker function.

3) Relation between 1) & 2).

1) Complex side.

Notations:

G reductive group$/\mathbb{C}$.

B - Borel subgp.

T - max. torus.

W - Weyl group

$X := G/B$ flag variety,

$w \in W$, $X(w)^\circ := BwB/B$ Schubert cell.

$X(w) := \overline{X(w)^\circ}$ Schubert variety.

- Definition of motivic Chern classes (K-theoretic generalization of MacPherson transformation).

two functors: $\cdot\ Y/\mathbb{C}$, $K^0(Var/Y) := \{[Z \xrightarrow{f} Y]\} \big/ [Z \xrightarrow{f} Y] = [U \xrightarrow{f} Y] + [Z \setminus U \xrightarrow{f} Y]$

$U \subseteq Z$ open.

$\cdot\ K(Y) := K^0(Coh(Y))$.

Thm (Brasselet-Schürmann-Yokura).

$\exists!$ natural transformation

$MC_y : K^0(Var/-) \to K(-)[y]$, s.t. if Y is smooth,

$MC_y([Y \xrightarrow{id} Y]) = \lambda_y(T^*Y) := \sum_i y^i [\Lambda^i T^*Y]$. Here y is a formal variable.

Remark: $\exists$ equivariant generalizations. Fehér-Rimányi-Weber, Aluffi-Mihalcea-Schürmann-S.

- Flag variety setting.

$T \curvearrowright X = G/B$. $\quad K_T(X) := K^\circ(\mathrm{Coh}_T(X))$.

$K_T(pt) = K^\circ(\mathrm{Coh}_T(pt)) = K^\circ(\mathrm{Rep}(T)) = \mathbb{Z}[T]$.

Let $MC_y(X(w)^\circ) := MC_y([X(w)^\circ \hookrightarrow X]) \in K_T(X)[y]$.

Ex. $G = SL(2,\mathbb{C})$, $X = \mathbb{P}^1$, $\quad MC_y(X(id)^\circ) = [\mathcal{O}_0]$

$MC_y(X(s_\alpha)^\circ) = MC_y(\mathbb{P}^1) - MC_y(X(id)^\circ) = \lambda_y(T^*\mathbb{P}^1) - [\mathcal{O}_0]$.

Demazure-Lusztig operator.

α_i: simple root, $\quad B \subseteq P_i$ minimal parabolic

$\pi_i : G/B \to G/P_i \qquad$ BGG operator (Demazure). $\quad \partial_i := \pi_i^* \pi_{i*} \curvearrowright K_T(X)$.

$\forall \lambda \in X^*(T), \quad \mathcal{L}_\lambda := G \times_B \mathbb{C}_\lambda$

$$\downarrow$$

$$X$$

Hecke algebra

Let $T_i = (1+y\mathcal{L}_{\alpha_i})\partial_i - \mathrm{id}$, $\quad (T_i+1)(T_i+y) = 0$, $\quad$ Braid relations.

Thm: (Aluffi-Mihalcea-Schurmann.-S)

1). $T_i(MC_y(X(w)^\circ)) = MC_y(X(ws_i)^\circ)$ if $ws_i > w$.

In particular, $(y=0, \rightsquigarrow$ ideal sheaf$)$

$$MC_y(X(w)^\circ) = T_{w^{-1}}([\mathcal{O}_{X(id)}])$$

2) $i: X \hookrightarrow T^*X$,

$$MC_y(X(w)^\circ) \text{"="} \, i^* \, gr[\mathbb{Q}^H_{X(w)^\circ}]$$

constant mixed Hodge module.

2) Langlands dual side.

- Principal series representation.

F non-archimedean local field. $\mathcal{O}_F \subseteq F$, k_F = residue field = $\mathbb{F}_q$, a finite field.

$G^\vee$ = Langlands dual group $/F$., $T^\vee \subseteq B^\vee \subseteq G^\vee$, I = Iwahori-subgroup.,
$$\begin{array}{ccc} I & \subseteq & G^\vee(\mathcal{O}_F) \\ \downarrow & & \downarrow \\ B^\vee(k_F) & \subseteq & G^\vee(k_F) \end{array}$$

τ — an unramified char. of $T^\vee$ ($\Leftrightarrow \tau \in T$)

Principal series $\mathrm{Ind}_{B^\vee_-(F)}^{G^\vee(F)}(\tau) \circlearrowleft G^\vee(F)$

Iwahori-Hecke alg.

Let $I(\tau) := \left(\mathrm{Ind}_{B^\vee_-(F)}^{G^\vee(F)}(\tau)\right)^I \circlearrowleft \mathbb{C}_c[I \backslash G^\vee(F) / I]$

A standard basis in $I(\tau)$:

$$\{\varphi_w \mid w \in W\}, \quad \varphi_w = \mathbb{1}_{B_-^\vee(F) w I}.$$

$$G^\vee(F) = \bigsqcup_{w \in W} B_-^\vee(F) w I.$$

- Iwahori-Whittaker functions.

σ – an unramified principal character of $N^\vee(F)$ $\quad B^\vee = T^\vee \cdot N^\vee$

Whittaker functional:

$$L : \mathrm{Ind}_{B_-^\vee(F)}^{G^\vee(F)} \tau \to \mathbb{C}, \quad \text{s.t.} \quad L(n\phi) = \sigma(n) \cdot L(\phi), \quad n \in N^\vee(F).$$

For any $f \in \mathrm{Ind}_{B_-^\vee(F)}^{G^\vee(F)} \tau$,

define $W_\tau(f) : G^\vee(F) \to \mathbb{C}$

$$g \mapsto L(g \cdot f).$$

inv. under the max. compact subgp. $G^\vee(\mathcal{O}_F)$

spherical vector.

Spherical Whittaker function. $W_\tau\left(\sum_\omega \varphi_\omega\right) : G^\vee(F) \to \mathbb{C}.$

Thm: (Casselman-Shalika formula.)

μ anti-dominant coweight of $G^\vee$, ϖ - uniformizer of $\mathcal{O}_F$

$$W_{\tau^{-1}}\left(\sum_\omega \varphi_\omega\right)(\varpi^{-\mu}) = (*) \prod_{\alpha>0}\left(1-q^{-1}e^{\alpha}(\tau)\right)\cdot \chi_{w_0\mu}(\tau)$$

char. of irr. highest weight $w_0\mu$ rep. of G.

Iwahori-Whittaker functions:

$$W_\tau(\varphi_\omega) : G^\vee(F) \to \mathbb{C}.$$

3) Relations between 1) and 2).

Borel-Weil $\Rightarrow$

$$\chi_{w_0\mu} = \chi(G/B, \mathcal{L}_\mu) := \sum_i (-1)^i H^i(G/B, \mathcal{L}_\mu) \in K_T(pt)$$

Casselman-Shalika $\Longrightarrow$

$$W_{\tau^{-1}}\left(\sum_w \varphi_w\right)(\varpi^{-\mu}) = (*) \prod_{\alpha>0}(1-q^{-1}e^{\alpha}(\tau)) \cdot \chi(G/B, \mathcal{L}_\mu)(\tau).$$

Thm: (Mihalcea-S.) μ anti-dominant coweight of $G^\vee$,

Segre-type class.

$$\mathcal{W}_{\tau^{-1}}(\varphi_w)(\varpi^{-\mu}) \overset{(*)}{=} \prod_{\alpha>0}(1-q^{-1}e^{\alpha}(\tau)) \cdot \chi\left(G/B, \mathcal{L}_\mu \otimes \frac{MC_{-q^{-1}}(X(w)^\circ)}{\lambda_{-q^{-1}}(T^*G/B)}\right)(\tau).$$

Remark: 1) Summing over $w \in W$, we get the Casselman-Shalika formula.

2). proof uses work of Brubaker-Bump-Licata.

3) can also be computed from colored vertex models (Brubaker-Buciumas-Bump-Gustafsson).

Thank you!