Equivariant K-theory of the semi-infinite flag manifold as a nil-DAHA module

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[O.] arXiv:2001.03490 [Kouno-Naito-O.-Sagaki] arXiv:2008.10483

(Inverse) Chevalley formula in $K_T(G/B)$

$X(w) = \overline{BwB/B} \subset G/B$
$\mathcal{O}(\mu) = G \times^B \mathbb{C}_{-\mu}$
$e^\lambda \in R(T)$

Chevalley formula [Pittie-Ram, Littelmann-Seshadri, Lenart-Postnikov, Griffeth-Ram]

$$[\mathcal{O}_{X(w)}(\mu)] = \sum_{\substack{v \in W \\ \lambda \in P}} c_{w,v}^{\mu,\lambda}\, e^\lambda \cdot [\mathcal{O}_{X(v)}] \qquad (c_{w,v}^{\mu,\lambda} \in \mathbb{Z})$$

Inverse Chevalley formula

$$e^\mu \cdot [\mathcal{O}_{X(w)}] = \sum_{\substack{v \in W \\ \lambda \in P}} d_{w,v}^{\mu,\lambda}\, [\mathcal{O}_{X(v)}(\lambda)] \qquad (d_{w,v}^{\mu,\lambda} \in \mathbb{Z})$$

These are one and the same

$$c_{w,v}^{\mu,\lambda} = d_{w^{-1},v^{-1}}^{-\mu,-\lambda}$$

Groups and such

G simply-connected, simple algebraic group over $\mathbb{C}$
$G = \sqcup_{w \in W} BwB$
$B = TU \subset G$

$G(\!(z)\!) = G(\mathbb{C}(\!(z)\!))$
$I = \mathrm{ev}_0^{-1}(B) \subset G[\![z]\!] = G(\mathbb{C}[\![z]\!])$
$Q^\vee = T(\!(z)\!)/T[\![z]\!]$

Iwasawa decomposition

$$G(\!(z)\!) = \bigsqcup_{\substack{w \in W \\ \xi \in Q^\vee}} I \cdot wz^\xi \cdot U(\!(z)\!)$$

Affine Weyl group

$$wz^\xi \in W \ltimes Q^\vee = W_{\mathrm{aff}}$$

Semi-infinite flag manifold of G

Definition (at the level of points) [Feigin-Frenkel]

$$\mathbf{Q}^{\mathrm{rat}} = \frac{G(\!(z)\!)}{T(\mathbb{C}) \cdot U(\!(z)\!)} = \bigsqcup_{\substack{w \in W \\ \xi \in Q^\vee}} I \cdot [wz^{\xi}]$$

[Finkelberg-Mirkovic]: $\mathbf{Q}^{\mathrm{rat}}$ is an ind-infinite scheme, via Plücker embedding into $\prod_{\lambda \in P_+} \mathbb{P}(V(\lambda)(\!(z)\!))$.

Semi-infinite Schubert varieties

For each $x = wz^{\xi} \in W_{\mathrm{aff}}$, let $\mathbf{Q}(x) = \overline{I \cdot [x]} = \bigsqcup_{y \succeq x} I \cdot [y] \subset \mathbf{Q}^{\mathrm{rat}}$.

- $\mathbf{Q}(x)$ infinite-dimensional *and* infinite-codimensional in $\mathbf{Q}^{\mathrm{rat}}$
- $\succeq$ = semi-infinite Bruhat order/Lusztig's generic Bruhat order

Some sources of motivation for studying $\mathbf{Q}^{\text{rat}}$

- Representations of affine Lie algebras [Feigin-Frenkel]
- Geometry of quasimaps $\mathbb{P}^1 \to G/B$ [Drinfeld, Finkelberg-Mirkovic]
- Quantum K-theory of G/B [Braverman-Finkelberg, Kato]

$$\boxed{K_T(\mathbf{Q}^{\text{rat}}) \cong qK_T(G/B)_{\text{loc}}} \qquad \text{(Kato's isomorphism)}$$

- Peterson's isomorphism and its extension to K-theory [Peterson, Lam-Shimozono, Lam-Li-Mihalcea-Shimozono, Kato]
- Combinatorics of level-zero (quantum) affine algebra representations [Kato-Naito-Sagaki, Feigin-Makedonskyi, Lenart-Naito-Sagaki]
- Geometric realizations of integrable systems
  - q-Toda [Givental-Lee, Braverman-Finkelberg]
  - (q, t)-Macdonald [Koroteev-Zeitlin]

Equivariant K-theory

$\mathbf{Q}^{\mathrm{rat}}$ is not Noetherian, which makes the usual approach to K-theory problematic.

[Kato-Naito-Sagaki] introduce $\boxed{K_{I\rtimes\mathbb{C}^*}(\mathbf{Q}^{\mathrm{rat}})}$ with good properties:

- Classes $[\mathcal{E}]$ for suitable quasi-coherent $\mathcal{E}$, including Schubert classes $[\mathcal{O}_{\mathbf{Q}(x)}]$ for $x \in W_{\mathrm{aff}}$
- Multiplication by equivariant scalars $\mathbb{Z}[P]((q^{-1})) \supset R(I \rtimes \mathbb{C}^\times)$
- Multiplication by equivariant line bundles $[\mathcal{O}(\lambda)]$ $(\lambda \in P)$

Combinatorial Chevalley formulas: $[\mathcal{O}_{\mathbf{Q}(x)}(\mu)]$ into $e^\lambda \cdot [\mathcal{O}_{\mathbf{Q}(y)}]$

- μ dominant [Kato-Naito-Sagaki] *infinite sums needed*
- μ anti-dominant [Naito-O.-Sagaki] *only finite sums*
- μ arbitrary [Lenart-Naito-Sagaki]

What about inverse Chevalley? It's not the same!

nil-DAHA on the left

Let $\mathbb{H}_{q,0}^X = \mathbb{Z}[q^{\pm 1}]\langle X^\mu, D_i \mid \mu \in P,\, i \in I_{\mathrm{aff}}\rangle \,/ \sim$ be the nil-DAHA (double affine Hecke algebra).

Theorem [Kato-Naito-Sagaki]

The algebra $\mathbb{H}_{q,0}^X$ acts on $K_{I \rtimes \mathbb{C}^*}(\mathbf{Q}^{\mathsf{rat}})$ from the left:

$$D_i \cdot [\mathcal{O}_{\mathbf{Q}(x)}] = \begin{cases} [\mathcal{O}_{\mathbf{Q}(x)}] & \text{if } s_i x \succ x \\ [\mathcal{O}_{\mathbf{Q}(s_i x)}] & \text{if } s_i x \prec x \end{cases}$$

$$X^\mu \cdot [\mathcal{O}_{\mathbf{Q}(x)}] = e^{-\mu} \cdot [\mathcal{O}_{\mathbf{Q}(x)}]$$

Note: This action includes equivariant scalar multiplication.

Heisenberg on the right

Let $\mathfrak{H}$ be the q-Heisenberg algebra generated by x^λ $(\lambda \in P)$, y^α $(\alpha \in Q^\vee)$ such that:

$$x^\lambda y^\alpha = q^{\langle \lambda, \alpha \rangle} y^\alpha x^\lambda.$$

Proposition (immediate from [Kato-Naito-Sagaki])

1. The algebra $\mathfrak{H}$ acts on $K_{I \rtimes \mathbb{C}^*}(\mathbf{Q}^{\mathrm{rat}})$ from the right:

$$[\mathcal{O}_{\mathbf{Q}(x)}(\mu)] \cdot x^\lambda = [\mathcal{O}_{\mathbf{Q}(x)}(\mu + \lambda)]$$
$$[\mathcal{O}_{\mathbf{Q}(x)}(\mu)] \cdot y^\alpha = q^{\langle \alpha, \mu \rangle} \cdot [\mathcal{O}_{\mathbf{Q}(xz^\alpha)}(\mu)].$$

2. The classes $[\mathcal{O}_{\mathbf{Q}(w)}]$ for $w \in W$ generate a free $\mathfrak{H}$-submodule.

Observation

The actions of $\mathbb{H}^X_{q,0}$ and $\mathfrak{H}$ commute.

Free $\mathfrak{H}$-submodule is a bimodule

Theorem [O.] – assume G simply-laced

The $\mathfrak{H}$-submodule gen. by $\{[\mathcal{O}_{\mathbf{Q}(w)}]\}_{w\in W}$ is stable under nil-DAHA $\mathbb{H}^X_{q,0}$. Equivalently, inverse Chevalley formula for $K_{I\rtimes\mathbb{C}^*}(\mathbf{Q}^{\mathrm{rat}})$ is *finite* (always).

Key Point: This gives $\mathbb{H}^X_{q,0} \xhookrightarrow{\rho_{\mathrm{geo}}} \mathrm{Mat}_W(\mathfrak{H})$.

Example: $G = SL(2)$

$$\rho_{\mathrm{geo}}(D_1) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \rho_{\mathrm{geo}}(D_0) = \begin{pmatrix} 0 & 0 \\ y^{-\alpha^\vee} & 1 \end{pmatrix}$$

$$\rho_{\mathrm{geo}}(X^{-\omega}) = \begin{pmatrix} x^{\omega} & x^{\omega}y^{\alpha^\vee} \\ -x^{\omega} & x^{-\omega} - x^{\omega}y^{\alpha^\vee} \end{pmatrix}$$

$$e^{\omega}\cdot[\mathcal{O}_{\mathbf{Q}(e)}] = [\mathcal{O}_{\mathbf{Q}(e)}(\omega)] - [\mathcal{O}_{\mathbf{Q}(s)}(\omega)]$$

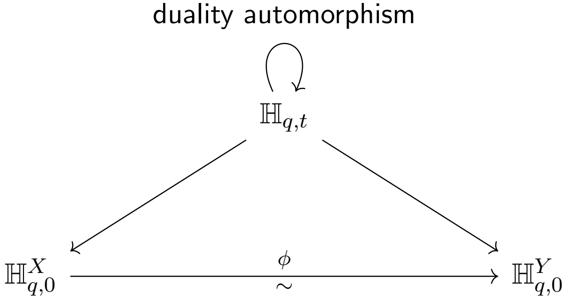
Question

What are images $\rho_{\mathrm{geo}}(X^{\mu})$? *These encode the inverse Chevalley formula.*

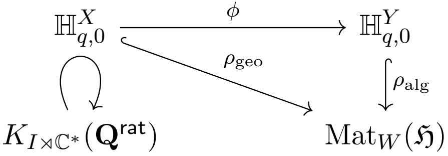
(nil-)DAHA duality – assume G simply-laced

DAHA $\mathbb{H}_{q,t} = \mathbb{Q}(q,t)\langle X^\lambda, T_w, Y^\mu \mid w \in W, \lambda, \mu \in P\rangle \,/\!\sim$

nil-DAHA's $\mathbb{H}^X_{q,0}\,,\ \mathbb{H}^Y_{q,0}$



(nil-)DAHA duality – assume G simply-laced



Theorem [O.]

There exists an explicit homomorphism ρ_{alg} making this diagram commute.

- *Explicit* means: to compute $\rho_{\text{geo}}(X^\mu)$, we take a reduced expression in the extended affine Weyl group and then build/manipulate an operator in the polynomial representation of $\mathbb{H}_{q,t}$.
- For specific μ (e.g., minuscule) this leads to QBG-based inverse Chevalley formulas [Kouno-Naito-O.-Sagaki].
- For arbitrary μ, can show agreement with (inverse) Chevalley formula in $K_T(G/B)$ [Lenart-Postnikov] via truncation.

Example: $G = SL(n+1)$

Theorem [Kouno-Naito-O.-Sagaki]

For $1 \leq i \leq n+1$, the inverse Chevalley product $e^{\varepsilon_i} \cdot [\mathcal{O}_{\mathbf{Q}(w_\circ)}]$ is given by

$$[\mathcal{O}_{\mathbf{Q}(w_\circ)}(-\varepsilon_i)] - \mathbf{1}_{\{i<n+1\}} \cdot q \cdot [\mathcal{O}_{\mathbf{Q}(w_\circ z^{-w_\circ(\alpha_i)})}(-\varepsilon_{i+1})]$$
$$+ \sum_{\varnothing \neq \{i_1 < \cdots < i_a\} \subset \{1,\ldots,i-1\}} (-1)^a [\mathcal{O}_{\mathbf{Q}((i_1 \cdots i_a i)^{-1} w_\circ z^{-w_\circ(\alpha_{i_1,i})})}(-\varepsilon_i)]$$
$$+ \sum_{\varnothing \neq \{j_1 < \cdots < j_b\} \subset \{i+1,n+1\}} (-1)^{b-1} q \cdot [\mathcal{O}_{\mathbf{Q}((i j_1 \cdots j_b)^{-1} w_\circ z^{-w_\circ(\alpha_{i,j_b})})}(-\varepsilon_{j_b})]$$

where

$$\mathbf{1}_{\{i<n+1\}} = \begin{cases} 1 & \text{if } i < n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Note: The class $[\mathcal{O}_{\mathbf{Q}(w_\circ)}]$ is an $(\mathbb{H}_{q,0}^X, \mathfrak{H})$-cyclic vector; hence this special case determines the entire inverse Chevalley formula.