Positivity of Chern and Segre MacPherson classes

Leonardo Mihalcea (Virginia Tech)

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Based on:

- P. Aluffi - M. - J. Schürmann - C. Su, *Positivity of Segre-MacPherson classes*, arχiv:1902.00762, to appear in special volume dedicated to W. Fulton 80th birthday.
- P. Aluffi - M. - J. Schürmann - C. Su, *Shadows of characteristic classes, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells.*, arχiv:1709.07106.

Euler characteristic

Let X be a compact manifold, T_X tangent bundle. Consider the Chern class

$$c(T_X) = 1 + c_1(T_X) + \ldots + c_n(T_X).$$

The topological Euler characteristic of X may be calculated from the Gauss-Bonnet Theorem:

$$c_n(T_X) \cap [X] = \chi(X)$$

Question: What happens if X is singular ?

Constructible functions

Let X be an algebraic variety. Constructible functions:

$$\mathcal{F}(X) = \{\sum c_i \mathbb{1}_{V_i} : c_i \in \mathbb{Z}, V_i \subset X \text{ constructible } \}.$$

If $f : X \to Y$ is a proper map, define a push-forward

$$f_* : \mathcal{F}(X) \to \mathcal{F}(Y); \quad f_*(\mathbb{1}_V)(y) = \chi(f^{-1}(y) \cap V).$$

Example

If $f : X \to pt$ (proper), then

$$f_*(\mathbb{1}_X) = \chi(X),$$

the topological Euler characteristic of X.

MacPherson's transformation

Theorem (Deligne - Grothendieck Conjecture; MacPherson '74, M. H. Schwartz '65)

There exists a unique natural transformation $c_* : \mathcal{F}(X) \to H_*(X)$ *such that:*

1. *If* X *is projective, non-singular,* $c_*(\mathbb{1}_X) = c(T_X) \cap [X]$.
2. c_* *is functorial with respect to proper push-forwards* $f : X \to Y$*:*

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{c_*} & H_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathcal{F}(Y) & \xrightarrow{c_*} & H_*(Y) \end{array}$$

Constructible functions $\rightsquigarrow$ characteristic classes of singular varieties:

- $\varphi = \mathbb{1}_U$ ($U \subset X$ constructible) $\rightsquigarrow$ Chern-Schwartz-MacPherson (CSM) class

$$c_{\mathrm{SM}}(U) \in H_*(X).$$

- If X-smooth, the Segre-MacPherson class is:

$$s_{\mathrm{M}}(U) = \frac{c_{\mathrm{SM}}(U)}{c(T_X)}.$$

Examples

1. $X = \mathbb{P}^1$. Then

$$c_{\mathrm{SM}}(\mathbb{P}^1) = c(T_{\mathbb{P}^1}) \cap [\mathbb{P}^1] = [\mathbb{P}^1] + 2[pt]; \quad s_{\mathrm{M}}(\mathbb{P}^1) = \frac{c(T_{\mathbb{P}^1}) \cap [\mathbb{P}^1]}{c(T_{\mathbb{P}^1})} = [\mathbb{P}^1].$$

2. $c_{\mathrm{SM}}[pt] = [pt]$ and $\mathbb{1}_{\mathbb{A}^1} = \mathbb{1}_{\mathbb{P}^1} - \mathbb{1}_{pt}$, thus

$$c_{\mathrm{SM}}(\mathbb{A}^1) = c_{\mathrm{SM}}(\mathbb{P}^1) - c_{\mathrm{SM}}(pt) = [\mathbb{P}^1] + [pt].$$

$$s_{\mathrm{M}}(\mathbb{A}^1) = \frac{c_{\mathrm{SM}}(\mathbb{P}^1) - c_{\mathrm{SM}}(pt)}{[\mathbb{P}^1] + 2[pt]} = [\mathbb{P}^1] - [pt].$$

3. CSM and SSM may be ‘different’:

$$c_{\mathrm{SM}}(\mathbb{A}^2) = [\mathbb{P}^2] + 2[\mathbb{P}]^1 + [pt].$$

$$s_{\mathrm{M}}(\mathbb{A}^2) = [\mathbb{P}^2] - [\mathbb{P}]^1 + [pt].$$

4. P. Aluffi: X-toric with open T-orbit X°, then

$$c_{\mathrm{SM}}(X^\circ) = [\overline{X^\circ}].$$

(Therefore $\chi(X^\circ) = 0$ unless $X^\circ = pt$.)

Schubert data

The flag manifold is

$$\mathrm{Fl}(n) = \{F_1 \subset F_2 \subset \ldots \subset \mathbb{C}^n\}.$$

It is homogeneous under $G := GL_n$, and has finitely many B-orbits. (B:= Borel subgroup of UT matrices.)

- Schubert varieties: X_w are indexed by permutations $w \in S_n$, and

$$X_w = \overline{X_w^\circ} = \overline{Be_w}; \quad X^w = \overline{X^{w,\circ}} = \overline{B^- e_w}$$

  where e_w is T-fixed point and B^- is the opposite Borel.
- Schubert classes give a basis for homology:

$$H_*(\mathrm{Fl}(n)) = \bigoplus_{w \in S_n} [X_w] = \bigoplus_{w \in S_n} [X^w].$$

We will work with (co)homology of generalized flag manifolds such as $\mathrm{Fl}(i_1, \ldots, i_k; n)$, or G/P.

Demazure-Lusztig operators

For $s_k \in W$ simple reflection let $s_k \in Aut(H^*(G/B))$ (the right Weyl group action), $\pi_k : \mathrm{Fl}(n) \to \mathrm{Fl}(\widehat{k}, n)$ (the projection) and

$$\partial_k = \pi_k^*(\pi_k)_* = \frac{1 - s_k}{x_k - x_{k+1}}$$

(the BGG operator). Define:

$$\mathcal{T}_k^{\pm} := \partial_k \pm s_k$$

(degenerate Demazure - Lusztig operator). It appears in the study of the degenerate Hecke algebra (Ginzburg, Lascoux-Leclerc-Thibon).

Lemma

The operators $\mathcal{T}_k^{\pm}$ satisfy the following properties:

1. **Commutativity**: *E.g. in type A,* $\mathcal{T}_i^{\pm}\mathcal{T}_j^{\pm} = \mathcal{T}_j^{\pm}\mathcal{T}_i^{\pm}$ *if* $|i-j| \geq 2$;
2. **Braid relations**: *E.g. in type A:* $\mathcal{T}_i^{\pm}\mathcal{T}_{i+1}^{\pm}\mathcal{T}_i^{\pm} = \mathcal{T}_{i+1}^{\pm}\mathcal{T}_i^{\pm}\mathcal{T}_{i+1}^{\pm}$;
3. **Square**: $(\mathcal{T}_i^{\pm})^2 = id$.
4. **Schubert action**: $\mathcal{T}_k^{-}([X(w)]) =$

$$\begin{cases} -[X_w] & \text{if } \ell(ws_k) < \ell(w) \\ [X_{ws_k}] + [X_w] + \sum \langle \alpha_k, \beta^\vee \rangle [X_{ws_k s_\beta}] & \text{if } \ell(ws_k) > \ell(w) \end{cases}$$

where $\beta > 0$, $\beta \neq \alpha_k$ *and* $\ell(ws_k s_\beta) = \ell(w)$.

CSM/SM classes and Cotangent Schubert Calculus

Theorem (Aluffi - M '16, AMSS '17)

Let $X = G/B$ and $w \in W$. Then the following hold:

1. **Hecke action**: $\mathcal{T}_i^- c_{SM}(X_w^\circ) = c_{SM}(X_{ws_i}^\circ)$; $\mathcal{T}_i^+ s_M(X_w^\circ) = s_M(X_{ws_i}^\circ)$.
2. **Stable envelopes**: *Let $stab_\pm(w) \subset T_X^*$ be the Maulik-Okounkov stable envelope, and let $\iota : X \to T^*X$ be the zero section. Then (Rimányi-Varchenko, AMSS)*
$$\iota^* stab_+(w) = \pm c_{SM}(X_w^\circ); \quad \iota^* stab_-(w) = \pm s_M(X_w^\circ).$$
3. **Cotagent Schubert Calculus**: *Let M_w be the Verma module from and $\mathrm{Char}(\mathcal{M}_w) \subset T_{G/P}^*$ its characteristic cycle. Then $\iota^*[\mathrm{Char}(\mathcal{M}_w)] = \pm c_{SM}(X_w^\circ)$.*
4. **Schubert basis**: *The CSM/SM classes deform the Schubert classes:*
$$c_{SM}(X_w^\circ) = [X_w] + \sum_{v<w} a_{w,v}[X_w].$$
5. **Poincaré duality**: $\langle c_{SM}(X_u^\circ), s_M(X^{v,\circ})\rangle = \delta_{u,v}$.
6. **Transversality** *(Schürmann): If $s_M(X^{u,\circ}) \cdot s_M(X^{v,\circ}) = \sum c_{u,v}^w s_M(X^{w,\circ})$ then*
$$c_{u,v}^w = \chi(g_1 X^{u,\circ} \cap g_2 X^{v,\circ} \cap g_3 X_w^\circ)$$
(topological Euler characteristic).

Positivity

Theorem (Huh '09 (Grassmannians); AMSS'17,'19)

Let $X = G/P$, and consider the Schubert expansions:

$$c_{SM}(X_w^\circ) = [X_w] + \sum_{v<w} a_{w,v}[X_w]; \quad s_M(X_w^\circ) = [X_w] + \sum_{v<w} b_{w,v}[X_w].$$

Then $a_{w,v} \geq 0$ and $(-1)^{\ell(w)-\ell(v)} b_{w,v} \geq 0$.

Alternation was conjectured by Feher-Rimányi '17 for Grassmanians.

Conjecture

Consider the expansion

$$s_M(X^{u,\circ}) \cdot s_M(X^{v,\circ}) = \sum c_{u,v}^{w} s_M(X^{w,\circ}).$$

Then $(-1)^{\ell(u)+\ell(v)+\ell(w)} c_{u,v}^{w} \geq 0$.

For k-step partial flag manifolds, $k \leq 3$, this is proved ('21) by Knutson and Zinn-Justin using puzzles. Their results extend to the K_T version, utilizing motivic Segre classes (Brasselet-Schürmann-Yokura '05, recent calculations by Maxim-Schürmann, Feher-Rimányi-Weber, AMSS, Anderson-Chen-Tarasca, ...).

The Lagrangian model and Segre classes

Let X complex, projective manifold and let $\mathbb{C}^*$ act on T^*_X with character $\hbar^{-1}$. Denote by $L_{\mathbb{C}^*}(X) \subset H^{\mathbb{C}^*}_*(T^*_X)$, the group of conic, Lagrangian cycles in T^*_X.

Example

If $Y \subset X$ closed, irreducible, its conormal space is:

$$T^*_Y X := \overline{T^*_{Y_{reg}} X} \subset T^*_X.$$

- The characteristic cycle map is:

$$CC : \mathcal{F}(X) \to L_{\mathbb{C}^*}(X); \quad Eu_Y \mapsto (-1)^{\dim Y}[T^*_Y(X)],$$

  where $Eu_Y(y)$ is MacPherson's local Euler obstruction of Y at y.
- If $C \subset T^*_X$ is a cone, and $q : \mathbb{P}(T^*_X \oplus \mathbb{1}) \to X$ is the projection, the Segre class is

$$Segre(C) := q_*\left(\frac{[\overline{C}]}{c(\mathcal{O}_{\mathbb{P}(T^*_X \oplus \mathbf{1})}(-1))}\right) = q_*\left(\sum_{i \geq 0} c_1(\mathcal{O}_{\mathbb{P}(T^*_X \oplus \mathbf{1})}(1))^i \cap [\overline{C}]\right) \in H_*(X).$$

Example

$Segre(T^*_X) = c(T^*_X)^{-1} \cap [X]$.

Positivity

Theorem (Sabbah '85, Ginzburg '86, Pragacz-Parusinski '01, Schürmann '05)

For $\varphi \in \mathcal{F}(X)$, *and let* $c_*(\varphi) = c_0 + c_1 + \ldots$, *where* $c_i \in A_i(X)$. *Define* $\check{c}_*(\varphi) = c_0 - c_1 + c_2 - \ldots$. *Then*

$$c(T_X^*) \cap Segre(CC(\varphi)) = \check{c}_*(\varphi).$$

Equivalently,

$$Segre(CC(\varphi)) = \frac{\check{c}_*(\varphi)}{c(T_X^*)}.$$

Lemma (Aluffi-M.-Schürmann-Su '17)

Let $\varphi = \mathbb{1}_{X_w^\circ}$. *Then the following hold:*
(a) $(-1)^{\ell(w)} CC(\mathbb{1}_{X_w^\circ})$ *is effective.*
(b) The Segre class $(-1)^{\ell(w)} Segre(CC(\mathbb{1}_{X_w^\circ}))$ *is an effective cycle in* $T_{G/P}^*$.

Proof.

Part (a) follows because $(-1)^{\ell(w)} CC(\mathbb{1}_{X_w^\circ})$ is the characteristic cycle of a holonomic $\mathcal{D}_{G/P}$-module (Brylinski-Kashiwara, Beilinson-Bernstein). Part (b) follows because $T_{G/P}$ is globally generated, therefore so is $\mathcal{O}_{\mathbb{P}(T_X^* \oplus \mathbf{1})}(1)$. Then its powers preserve effective cycles. □

Positivity (cont.)

Recall

$$(-1)^{\ell(w)} Segre(CC(\mathbb{1}_{X_w^\circ})) = \frac{\check{c}_*(\mathbb{1}_{X_w^\circ})}{c(T_X^*)} \geq 0.$$

Corollary

Let $X = G/P$.

(a) The Segre class $\frac{c_*(\mathbf{1}_{X_w^\circ})}{c(T_X)}$ is alternating.

(b) Let $P = B$. Then the CSM class $c_{SM}(X_w^\circ)$ is effective.

(c) For any $X_w^\circ \subset G/P$, the CSM class $c_{SM}(X_w^\circ)$ is effective.

Proof.

- Part (a) is a consequence of the previous Lemma.
- Part (b) follows because CSM and SM classes for $X_w^\circ \subset G/B$ differ by changing signs in homogeneous components (using the Hecke action).
- Part (c) follows by functoriality of CSM classes.

□

Remark. The alternation of $s_M(U)$ holds for any affine inclusions $U \hookrightarrow X$ such that T_X is globally generated.

THANK YOU!

CSM classes in $\mathrm{Gr}(2,5)$

	$(3,3)$	$(3,2)$	$(3,1)$	(3)	$(2,2)$	$(2,1)$	(2)	$(1,1)$	(1)	$\emptyset$
$\emptyset$										1
(1)									1	+ 1
$(1,1)$								1	+ 2	+ 1
(2)							1		+ 2	+ 1
$(2,1)$						1	+ 2	+ 2	+ 3	+ 1
$(2,2)$					1	+ 3	+ 4	+ 4	+ 4	+ 1
(3)				1			+ 3		+ 3	+ 1
$(3,1)$			1	+ 2		+ 3	+ 5	+ 3	+ 4	+ 1
$(3,2)$		1	+ 3	+ 4	+ 3	+ 8	+ 8	+ 6	+ 5	+ 1
$(3,3)$	1	+ 4	+ 7	+ 8	+ 8	+15	+12	+ 9	+ 6	+ 1

Examples for $\mathrm{Fl}(3)$

Recall that $H^*(G/B) = \oplus_{w \in W} \mathbb{Z}[X(w)]$. Then:

$$c_{\mathrm{SM}}(X(w)^o) = \sum_{v \leq w} c(w;v)[X(v)] = 1 \cdot [X(w)] + \ldots + 1 \cdot [pt].$$

Consider the flag variety $\mathrm{Fl}(3) = \{F_1 \subset F_2 \subset \mathbb{C}^3\}$.

1. $c_{\mathrm{SM}}(X(s_1)^\circ) = \mathcal{T}_1(c_{\mathrm{SM}}[pt]) = (\partial_1 - s_1)[pt] = [X(s_1)] + [pt]$. (Recall $X(s_1)^\circ \simeq \mathbb{P}^1 \setminus pt$.)
2. The CSM of the open Schubert cell $c_{\mathrm{SM}}(\mathrm{Fl}(3)^o) = c_{\mathrm{SM}}(X(s_1 s_2 s_1)^\circ)$ is:
$$[\mathrm{Fl}(3)] + [X(s_2 s_1] + [X(s_1 s_2)] + 2[X(s_1)] + 2[X(s_2)] + [pt].$$
3. The total Chern class of $\mathrm{Fl}(3)$ is:
$$c(T_{\mathrm{Fl}(3)}) = \sum_{w \in S_3} c_{\mathrm{SM}}(X(w)^\circ)$$
$$= [\mathrm{Fl}(3)] + 2[X(s_2 s_1)] + 2[X(s_1 s_2)] + 6[X(s_1)] + 6[X(s_2)] + 6[pt].$$