Positivity of Chern and Segre MacPherson classes

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Special Session on Recent Advances in Schubert Calculus and Related Topics
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Based on:

Euler characteristic

Let $X$ be a compact manifold, $T_X$ tangent bundle. Consider the Chern class

$$c(T_X) = 1 + c_1(T_X) + \ldots + c_n(T_X).$$

The topological Euler characteristic of $X$ may be calculated from the Gauss-Bonnet Theorem:

$$c_n(T_X) \cap [X] = \chi(X)$$

**Question:** What happens if $X$ is singular?
Constructible functions

Let $X$ be an algebraic variety. **Constructible functions:**

$$\mathcal{F}(X) = \left\{ \sum c_i \mathbbm{1}_{V_i} : c_i \in \mathbb{Z}, V_i \subset X \text{ constructible} \right\}.$$ 

If $f : X \rightarrow Y$ is a proper map, define a push-forward

$$f_* : \mathcal{F}(X) \rightarrow \mathcal{F}(Y); \quad f_*(1 \mathbbm{1}_V)(y) = \chi(f^{-1}(y) \cap V).$$ 

**Example**

If $f : X \rightarrow pt$ (proper), then

$$f_*(1_X) = \chi(X),$$

the topological Euler characteristic of $X$. 
MacPherson’s transformation

Theorem (Deligne - Grothendieck Conjecture; MacPherson ’74, M. H. Schwartz ’65)

There exists a unique natural transformation $c_* : \mathcal{F}(X) \rightarrow H_*(X)$ such that:

1. If $X$ is projective, non-singular, $c_*(1_X) = c(T_X) \cap [X]$.
2. $c_*$ is functorial with respect to proper push-forwards $f : X \rightarrow Y$:

$$
\begin{array}{ccc}
\mathcal{F}(X) & \xrightarrow{c_*} & H_*(X) \\
\downarrow f_* & & \downarrow f_* \\
\mathcal{F}(Y) & \xrightarrow{c_*} & H_*(Y)
\end{array}
$$

Constructible functions $\rightsquigarrow$ characteristic classes of singular varieties:

- $\varphi = 1_U \ (U \subset X \text{ constructible}) \rightsquigarrow$ Chern-Schwartz-MacPherson (CSM) class

$$
c_{SM}(U) \in H_*(X).
$$

- If $X$-smooth, the Segre-MacPherson class is:

$$
s_{M}(U) = \frac{c_{SM}(U)}{c(T_X)}.
$$
Examples

1. $X = \mathbb{P}^1$. Then

$$c_{SM}(\mathbb{P}^1) = c(T_{\mathbb{P}^1}) \cap [\mathbb{P}^1] = [\mathbb{P}^1] + 2[pt]; \quad s_{M}(\mathbb{P}^1) = \frac{c(T_{\mathbb{P}^1}) \cap [\mathbb{P}^1]}{c(T_{\mathbb{P}^1})} = [\mathbb{P}^1].$$

2. $c_{SM}[pt] = [pt]$ and $1_{\mathbb{A}^1} = 1_{\mathbb{P}^1} - 1_{pt}$, thus

$$c_{SM}(\mathbb{A}^1) = c_{SM}(\mathbb{P}^1) - c_{SM}(pt) = [\mathbb{P}^1] + [pt].$$

$$s_{M}(\mathbb{A}^1) = \frac{c_{SM}(\mathbb{P}^1) - c_{SM}(pt)}{[\mathbb{P}^1] + 2[pt]} = [\mathbb{P}^1] - [pt].$$

3. CSM and SSM may be ‘different’:

$$c_{SM}(\mathbb{A}^2) = [\mathbb{P}^2] + 2[\mathbb{P}]^1 + [pt].$$

$$s_{M}(\mathbb{A}^2) = [\mathbb{P}^2] - [\mathbb{P}]^1 + [pt].$$

4. P. Aluffi: $X$-toric with open $T$-orbit $X^\circ$, then

$$c_{SM}(X^\circ) = [\overline{X^\circ}].$$

(Therefore $\chi(X^\circ) = 0$ unless $X^\circ = pt.$)
Schubert data

The flag manifold is

\[ \text{Fl}(n) = \{ F_1 \subset F_2 \subset \ldots \subset \mathbb{C}^n \}. \]

It is homogeneous under \( G := GL_n \), and has finitely many \( B \)-orbits. (\( B := \) Borel subgroup of UT matrices.)

- **Schubert varieties**: \( X_w \) are indexed by permutations \( w \in S_n \), and

\[ X_w = X_w^o = Be_w, \quad X^w = X_w^{o,} = B^- e_w \]

where \( e_w \) is \( T \)-fixed point and \( B^- \) is the opposite Borel.

- **Schubert classes** give a basis for homology:

\[ H_*(\text{Fl}(n)) = \bigoplus_{w \in S_n} [X_w] = \bigoplus_{w \in S_n} [X^w]. \]

We will work with (co)homology of generalized flag manifolds such as \( \text{Fl}(i_1, \ldots, i_k; n) \), or \( G/P \).
Demazure-Lusztig operators

For $s_k \in W$ simple reflection let $s_k \in \text{Aut}(H^*(G/B))$ (the right Weyl group action), $\pi_k : \text{Fl}(n) \to \text{Fl}(\hat{k}, n)$ (the projection) and

$$\partial_k = \pi_k^*(\pi_k)_* = \frac{1 - s_k}{x_k - x_{k+1}}$$

(the BGG operator). Define:

$$T_k^\pm := \partial_k \pm s_k$$

(degenerate Demazure - Lusztig operator). It appears in the study of the degenerate Hecke algebra (Ginzburg, Lascoux-Leclerc-Thibon).

Lemma

The operators $T_k^\pm$ satisfy the following properties:

- **Commutativity**: E.g. in type A, $T_i^\pm T_j^\pm = T_j^\pm T_i^\pm$ if $|i - j| \geq 2$;
- **Braid relations**: E.g. in type A: $T_i^\pm T_{i+1}^\pm T_i^\pm = T_{i+1}^\pm T_i^\pm T_{i+1}^\pm$;
- **Square**: $(T_i^\pm)^2 = id$.
- **Schubert action**: $T_k^-([X(w)]) =$

\[
\begin{cases} 
  -[X_w] & \text{if } \ell(ws_k) < \ell(w) \\
  [X_{ws_k}] + [X_w] + \sum \langle \alpha_k, \beta^\vee \rangle [X_{ws_k}s_\beta] & \text{if } \ell(ws_k) > \ell(w)
\end{cases}
\]

where $\beta > 0$, $\beta \neq \alpha_k$ and $\ell(ws_k s_\beta) = \ell(w)$. 
Let $X = G/B$ and $w \in W$. Then the following hold:

1. **Hecke action**: $T_i^- c_{SM}(X_w^\circ) = c_{SM}(X_{ws_i}^\circ)$; $T_i^+ s_{SM}(X_w^\circ) = s_{SM}(X_{ws_i}^\circ)$.

2. **Stable envelopes**: Let $\text{stab}_{\pm}(w) \subset T^*_X$ be the Maulik-Okounkov stable envelope, and let $\iota : X \rightarrow T^* X$ be the zero section. Then (Rimányi-Varchenko, AMSS)
   
   \[ \iota^* \text{stab}_+ (w) = \pm c_{SM}(X_w^\circ); \quad \iota^* \text{stab}_- (w) = \pm s_{SM}(X_w^\circ). \]

3. **Cotagent Schubert Calculus**: Let $M_w$ be the Verma module from and $\text{Char}(M_w) \subset T^*_{G/P}$ its characteristic cycle. Then $\iota^* [\text{Char}(M_w)] = \pm c_{SM}(X_w^\circ)$.

4. **Schubert basis**: The CSM/SM classes deform the Schubert classes:

   \[ c_{SM}(X_w^\circ) = [X_w] + \sum_{v < w} a_{w,v} [X_w]. \]

5. **Poincaré duality**: \( \langle c_{SM}(X_u^\circ), s_M(X_v^\circ) \rangle = \delta_{u,v} \).

6. **Transversality (Schürmann)**: If $s_M(X_u^\circ) \cdot s_M(X_v^\circ) = \sum c_{u,v}^w s_{SM}(X_w^\circ)$ then

   \[ c_{u,v}^w = \chi(g_1 X_u^\circ \cap g_2 X_v^\circ \cap g_3 X_w^\circ) \]

   (topological Euler characteristic).
Theorem (Huh ’09 (Grassmannians); AMSS’17,’19)

Let $X = G/P$, and consider the Schubert expansions:

\[
\begin{align*}
    c_{SM}(X_w^\circ) &= [X_w] + \sum_{v < w} a_{w,v} [X_w]; \\
    s_{M}(X_w^\circ) &= [X_w] + \sum_{v < w} b_{w,v} [X_w].
\end{align*}
\]

Then $a_{w,v} \geq 0$ and $(-1)^{\ell(w)-\ell(v)} b_{w,v} \geq 0$.

Alternation was conjectured by Feher-Rimányi ’17 for Grassmanians.

Conjecture

Consider the expansion

\[
    s_{M}(X_u^\circ) \cdot s_{M}(X_v^\circ) = \sum c_{u,v}^w s_{M}(X_w^\circ).
\]

Then $(-1)^{\ell(u)+\ell(v)+\ell(w)} c_{u,v}^w \geq 0$.

For $k$-step partial flag manifolds, $k \leq 3$, this is proved (‘21) by Knutson and Zinn-Justin using puzzles. Their results extend to the $K_T$ version, utilizing motivic Segre classes (Brasselet-Schürmann-Yokura ’05, recent calculations by Maxim-Schürmann, Feher-Rimányi-Weber, AMSS, Anderson-Chen-Tarasca, ...).
The Lagrangian model and Segre classes

Let $X$ complex, projective manifold and let $\mathbb{C}^*$ act on $T_X^*$ with character $\hbar^{-1}$. Denote by $L_{\mathbb{C}^*}(X) \subset H_*^{\mathbb{C}^*}(T_X^*)$, the group of conic, Lagrangian cycles in $T_X^*$.

Example

If $Y \subset X$ closed, irreducible, its conormal space is:

$$T_Y^*X := \overline{T_Y^{\text{reg}}X} \subset T_X^*.$$ 

- The characteristic cycle map is:

$$CC : \mathcal{F}(X) \to L_{\mathbb{C}^*}(X); \quad Eu_Y \mapsto (-1)^{\dim Y}[T_Y^*(X)],$$

where $Eu_Y(y)$ is MacPherson’s local Euler obstruction of $Y$ at $y$.

- If $C \subset T_X^*$ is a cone, and $q : \mathbb{P}(T_X^* \oplus 1) \to X$ is the projection, the Segre class is

$$\text{Segre}(C) := q_*\left(\frac{[C]}{c(\mathcal{O}_{\mathbb{P}(T_X^* \oplus 1)}(-1))}\right) = q_*\left(\sum_{i \geq 0} c_1(\mathcal{O}_{\mathbb{P}(T_X^* \oplus 1)}(1))^i \cap [C]\right) \in H_*(X).$$

Example

$$\text{Segre}(T_X^*) = c(T_X^*)^{-1} \cap [X].$$
Theorem (Sabbah ’85, Ginzburg ’86, Pragacz-Parusinski ’01, Schüermann ’05)

For $\varphi \in \mathcal{F}(X)$, and let $c_*(\varphi) = c_0 + c_1 + \ldots$, where $c_i \in A_i(X)$. Define $\check{c}_*(\varphi) = c_0 - c_1 + c_2 - \ldots$. Then

$$c(T_X^*) \cap \text{Segre}(\text{CC}(\varphi)) = \check{c}_*(\varphi).$$

Equivalently,

$$\text{Segre}(\text{CC}(\varphi)) = \frac{\check{c}_*(\varphi)}{c(T_X^*)}.$$

Lemma (Aluffi-M.-Schüermann-Su ’17)

Let $\varphi = 1_{X_0^w}$. Then the following hold:

(a) $(-1)^{\ell(w)} \text{CC}(1_{X_0^w})$ is effective.

(b) The Segre class $(-1)^{\ell(w)} \text{Segre}(\text{CC}(1_{X_0^w}))$ is an effective cycle in $T_{G/P}^*$.

Proof.

Part (a) follows because $(-1)^{\ell(w)} \text{CC}(1_{X_0^w})$ is the characteristic cycle of a holonomic $\mathcal{D}_{G/P}$-module (Brylinski-Kashiwara, Beilinson-Bernstein). Part (b) follows because $T_{G/P}$ is globally generated, therefore so is $\mathcal{O}_P(T_X^* \oplus 1)(1)$. Then its powers preserve effective cycles.
Recall

\((-1)^{\ell(w)} \text{Segre}(CC(1_{X_w})) = \frac{\check{c}_*(1_{X_w})}{c(T^*_X)} \geq 0.\)

**Corollary**

Let \(X = G/P.\)

(a) The Segre class \(\frac{c_*(1_{X_w})}{c(T^*_X)}\) is alternating.

(b) Let \(P = B.\) Then the CSM class \(c_{SM}(X_w)\) is effective.

(c) For any \(X_w \subset G/P,\) the CSM class \(c_{SM}(X_w)\) is effective.

**Proof.**

- Part (a) is a consequence of the previous Lemma.
- Part (b) follows because CSM and SM classes for \(X_w \subset G/B\) differ by changing signs in homogeneous components (using the Hecke action).
- Part (c) follows by functoriality of CSM classes.

**Remark.** The alternation of \(s_{SM}(U)\) holds for any affine inclusions \(U \hookrightarrow X\) such that \(T_X\) is globally generated.
THANK YOU!
CSM classes in $\text{Gr}(2,5)$

\[
\begin{array}{ccccccccc}
 & & & & & & & & 1 \\
 & & & & & & & +1 \\
 & & & & & & 1 & +1 \\
 & & & & & 1 & +2 & +1 \\
 & & & & 1 & +2 & +2 & +3 & +1 \\
 & & & 1 & +3 & +4 & +4 & +4 & +1 \\
 & & 1 & +3 & +3 & +3 & +1 \\
 & 1 & +2 & +5 & +3 & +4 & +1 \\
 1 & +3 & +4 & +8 & +8 & +6 & +5 & +1 \\
 1 & +4 & +7 & +8 & +8 & +15 & +12 & +9 & +6 & +1 \\
\end{array}
\]
Examples for Fl(3)

Recall that $H^*(G/B) = \bigoplus_{w \in W} \mathbb{Z}[X(w)]$. Then:

$$c_{SM}(X(w)^\circ) = \sum_{\nu \leq w} c(w; \nu)[X(\nu)] = 1 \cdot [X(w)] + \ldots + 1 \cdot [pt].$$

Consider the flag variety $Fl(3) = \{ F_1 \subset F_2 \subset \mathbb{C}^3 \}$.

1. $c_{SM}(X(s_1)^\circ) = T_1(c_{SM}[pt]) = (\partial_1 - s_1)[pt] = [X(s_1)] + [pt].$
   (Recall $X(s_1)^\circ \simeq \mathbb{P}^1 \setminus pt.$)

2. The CSM of the open Schubert cell $c_{SM}(Fl(3)^\circ) = c_{SM}(X(s_1 s_2 s_1)^\circ)$ is:
   $$[Fl(3)] + [X(s_2 s_1)] + [X(s_1 s_2)] + 2[X(s_1)] + 2[X(s_2)] + [pt].$$

3. The total Chern class of Fl(3) is:
   $$c(T_{Fl(3)}) = \sum_{w \in S_3} c_{SM}(X(w)^\circ)$$
   $$= [Fl(3)] + 2[X(s_2 s_1)] + 2[X(s_1 s_2)] + 6[X(s_1)] + 6[X(s_2)] + 6[pt].$$