hli29@albany.edu

Equivariant oriented cohomology of Bott-Samelson varieties

Hao Li

SUNY at Albany

21 March

Overview

- Equivariant oriented cohomology theory
- Bott-Samelson varieties
- Restriction formulas and some applications

Equivariant oriented cohomology theory

Definition

An equivariant oriented cohomology theory over k is an additive contravariant functor h_G from the category G-Var of G-equivariant smooth quasi-projective varieties over k to the commutative rings with unit together with some axioms including

- a natural transformation of functors $c^G : K_G \to \tilde{h_G}$ ($\tilde{h_G}$ is total equivariant characteristic class).
- (Quillen's formula) If $\mathcal{L}_1$ and $\mathcal{L}_2$ are locally free sheaves of rank 1, then

$$c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) +_F c_1(\mathcal{L}_2),$$

  where F is the formal group law of h.

Equivariant oriented cohomology of a point

Let T be a split torus and Λ be the group of characters of T.

Consider the formal group algebra $R[\![\Lambda]\!]_F$, which is topologically generated by elements of form x_λ, $\lambda \in \Lambda$, which satisfy $x_{\lambda+\mu} = x_\lambda +_F x_\mu$.

Theorem (Calmès-Petrov-Zainoulline)

If h is (separated and) Chern complete over the point for T, then the natural map $h_T(pt) \to R[\![\Lambda]\!]_F$ is an isomorphism. It sends the characteristic class $c_1^T(\mathcal{L}_\lambda) \in h_T(pt)$ to $x_\lambda \in R[\![\Lambda]\!]_F$.

Example

- The equivariant Chow ring: $S_{\mathbb{Z}}(\Lambda)^\wedge$.
- The (completed) equivariant K-theory: $\mathbb{Z}[\Lambda]^\wedge$.
- The equivariant algebraic cobordism: $\mathbb{L}[\![\Lambda]\!]_U$.

Equivariant oriented cohomology of T-fixed points

Let G be a split algebraic group over k containing T as the maximal torus, with character group Λ. Let W be the Weyl group associated to (G, T). We denote the roots of G by Σ and choose a Borel subgroup B containing T.

The T-fixed k-points of G/B are in bijection with elements of W.

We can define an R-module $S_W := S \otimes_R R[W]$ with the product structure

$$q\delta_w q'\delta_{w'} = qw(q')\delta_{ww'}, \quad q, q' \in S, \quad w, w' \in W.$$

And we have

$$Hom_S(S_W, S) \cong h_T((G/B)^T),$$

where f_w is the dual basis of δ_w satisfying $f_w f_{w'} = \delta_{w,w'} f_w$.

Formal Demazure algebra

Let $Q_W = S[\frac{1}{x_\alpha} | \alpha \in \Sigma] \otimes_S S_W$, inside which we can define formal Demazure element:

$$X_\alpha = \frac{1}{x_\alpha} - \frac{1}{x_\alpha}\delta_{s_\alpha}.$$

The formal Demazure algebra $\mathcal{D}$ is the R-subalgebra of Q_W generated by elements from S and elements X_α, $\alpha \in \Sigma$.

Theorem (Calmes-Zainoulline-Zhong)

The pull-back map to fixed points $\imath^* : h_T(G/B) \to h_T(W)$ is injective, and its image is isomorphic to $Hom_S(\mathcal{D}, S)$.

Bott-Samelson varieties

Let P_i be a minimal parabolic subgroup corresponding to a simple root α_i.

Definition

For an l-tuple of integers $I = (i_1, i_2, \cdots, i_l)$ with $1 \leq i_j \leq n$, we define a variety $\hat{X}_I$ to be the fiber product

$$\hat{X}_I = P_{i_1} \times^B P_{i_2} \times^B \cdots \times^B P_{i_l}/B.$$

The multiplication all all factors induces a map $q_I : \hat{X}_I \to G/B$, which provides us a resolution of Shubert variety X_I if I is a reduced decomposition of $w(I) = s_{i_1} s_{i_2} \cdots s_{i_l}$.

The Bott-Salmelson class ζ_I is the push-forward $q_{I*}(1)$ in $h_T(G/B)$. For any choice of reduced sequence $\{I_w\}_{w \in W}$, the classes ζ_I generate $h_T(G/B)$ as an S-module.

Bott-Samelson varieties

Theorem (Calmes-Petrov-Zainoulline)

We have the following presentation

$$h_T(\hat{X}_I) \cong h_T(pt)[\eta_1, \eta_2, \cdots, \eta_l]/(\left\{{\eta_j}^2 - y_j\eta_j | j = 1, \cdots, l\right\}),$$

where

$$y_j = p^* \boldsymbol{c}_{(i_1,\ldots,i_{j-1})}(x_{-\alpha_{i_j}}), \quad \eta_j = p^* {\sigma_j}_*(1),$$

with p^* the pull-back from $h_T(\hat{X}_{(i_1,\ldots,i_j)})$ to $h_T(\hat{X}_I)$.

For each subset $L \in [l]$, define

$$\eta_L = \prod_{j \in L} \eta_j \in h_T(\hat{X}_I).$$

The S-module $h_T(\hat{X}_I)$ is free with basis $\{\eta_L | L \in \mathcal{P}_l\}$.

Bott-Samelson varieties

For $SL(4)$ whose simple roots are $\alpha_1, \alpha_2, \alpha_3$, let us consider Bott-Salmelson $\hat{X}_I = P_{_1} \times^B P_{_2} \times^B P_{_3}/B$. Then $h_T(\hat{X}_I)$ is a polynomial algebra generated by η_1, η_2, η_3 with following quotient relations:

$$
\begin{aligned}
&\eta_1^2 = x_{-\alpha_1}\eta_1, \\
&\eta_2^2 = x_{-\alpha_1-\alpha_2}\eta_1 + \frac{x_{-\alpha_2} - x_{\alpha_1-\alpha_2}}{x_{-\alpha_1}}\eta_1\eta_2, \\
&\eta_3^2 = x_{\alpha_1-\alpha_2-\alpha_3}\eta_3 + \frac{x_{-\alpha_3-\alpha_2} - x_{2\alpha_1-\alpha_2-\alpha_3}}{x_{-\alpha_1}}\eta_1\eta_3 + \frac{x_{\alpha_3} - x_{\alpha_1+\alpha_2-\alpha_3}}{x_{-\alpha_1-\alpha_2}}\eta_2\eta_3 \\
&+ \left(\frac{x_{-\alpha_3 - x_{\alpha_2-\alpha_3}}}{x_{-\alpha_2}x_{-\alpha_1}} - \frac{x_{-\alpha_3} - x_{\alpha_2-\alpha_1-\alpha_3}}{x_{\alpha_1-\alpha_2}x_{-\alpha_1}}\right)\eta_1\eta_2\eta_3.
\end{aligned}
$$

Bott-Samelson varieties

Lemma (Willems)

1. *The set $\hat{X}_I^T$ of T-fixed points in $\hat{X}_I$, consists of 2^l points*

$$[g_1, g_2, \cdots, g_l]$$

where $g_j \in \left\{e, s_{i_j}\right\}$. Here we think of s_{i_j} as in $W \cong N_G(T)/T$ and pick a preimage for s_{i_j} in $N_G(T) \subset G$. Consequently, we have bijection of sets from the power set $\mathcal{P}_l := \mathcal{P}([l])$ to $\hat{X}_I^T$,

$$L \mapsto pt_L := [g_1, ..., g_l], \quad g_j = \begin{cases} s_{i_j}, & \text{if } j \in L, \\ e, & \text{if } j \notin L. \end{cases}$$

2. *The set $(\hat{X}_I)_L$ is a T-orbit containing the fixed point pt_L, and isomorphic to the affine space of dimension $|L|$. The variety $\hat{X}_I$ has a decomposition $\coprod_{L \in \mathcal{E}_l} (\hat{X}_I)_L$.*

3. Suppose $L, L' \subset [l]$. then $pt_L \in \overline{(\hat{X}_I)_{L'}}$ if and only if $L \subset L'$. The weights of the T-action on the tangent space of $\overline{(\hat{X}_I)_{L'}}$ at pt_L are

$$\{-v_j^L(\alpha_{i_j}) | j \in L'\}.$$

Example

For the A_2-case, consider $\hat{X}_{(1,2)} = P_1 \times^B P_2/B$. There are four T-fixed points, denoted by $\{00, 01, 10, 11\}$, corresponding to $\{[e,e],[e,s_2],[s_1,e],[s_1,s_2]\}$, or $\emptyset, \{2\}, \{1\}, \{1,2\}$ as subsets of $[2]$. The weights of the tangent spaces of $\hat{X}_{(1,2)}$ at the four points are:

$$\begin{array}{llll} 00: & -\alpha_1, -\alpha_2 & 01: & -\alpha_1, \alpha_2 \\ 10: & \alpha_1, -\alpha_1-\alpha_2 & 11: & \alpha_1, \alpha_1+\alpha_2. \end{array}$$

Bott-Samelson varieties

We denote the set of functions on $\mathcal{E}_I = \hat{X}_I^T$ with values in S by $F(\mathcal{E}_I; S)$. It is a free S-module with basis $f_L, L \in \mathcal{E}_I$ defined by $f_L(L') = \delta_{L,L'}$, and have a ring structure given by $f_L \cdot f_{L'} = \delta_{L,L'} f_L$. Moreover, we have $h_T((\hat{X}_I)^T) \cong F(\mathcal{E}_I; S)$.

Theorem (L.-Zhong)

Let I be a sequence of length l. For any two subsets $L, M \subset [l]$ denote $L^c = [l] \backslash L$ and

$$a_{L,M} = \prod_{k \in L} v_{k-1}^M (x_{-\alpha_{i_k}}),$$

where $v_j^M = \prod_{k \in L \cap [j]} s_{i_k}$ Then

$$\mathbf{j}^*(\eta_L) = \sum_{M \subset L^c} a_{L,M} f_M.$$

The map $\mathbf{j}^* : h_T(\hat{X}_I) \to h_T(\hat{X}_I^T)$ is an injection.

Bott-Samelson varieties

Example

Consider the case of A_2. Let $\{\alpha_1, \alpha_2\}$ be the set of simple roots. We consider the Bott-Samelson variety $\hat{X}_I = P_1 \times^B P_2/B$ for $I = (1,2)$. There are four torus-fixed points, denoted by $\mathcal{P}_2 = \{00, 01, 10, 11\}$. Similarly, denote $(P_1/B)^T$ by $\mathcal{P}_1 = \{0, 1\}$. We have the following commutative diagram:

$$
\begin{array}{ccc}
P_1 \times^B P_2/B & \xleftarrow[\mathbf{j}^I]{} & \mathcal{P}_2 = \{00, 01, 10, 11\} . \\
\sigma_2 \uparrow \downarrow p_2 & & \downarrow p_2' \\
P_1/B & \xleftarrow[\mathbf{j}^1]{} & \mathcal{P}_1 = \{0, 1\} \\
\sigma_1 \uparrow \downarrow p_1 & & \\
pt & &
\end{array}
$$

Corollary

Example

We have

$$(\mathbf{j}^I)^*(\eta_1) = x_{-\alpha_1}(f_{00} + f_{01}).$$
$$(\mathbf{j}^I)^*(\eta_2) = x_{-\alpha_2} f_{00} + x_{-\alpha_1-\alpha_2} f_{10}.$$
$$(\mathbf{j}^I)^*(\eta_1\eta_2) = x_{-\alpha_1} x_{-\alpha_2} f_{00}.$$

In K-theory, $\eta_1 = [\overline{(\hat{X}_I)_{01}}]$, $\eta_2 = [\overline{(\hat{X}_I)_{10}}]$, $\eta_1\eta_2 = [\overline{(\hat{X}_I)_{00}}]$.
In general, we have

$$\eta_L = [\overline{(\hat{X}_I)_{L^c}}].$$

Bott-Samelson varieties

Theorem (L.-Zhong)

If $I = (i_1, ..., i_l)$ with i_j all distinct, then we have

$$im(\mathbf{j}^*) = \{\sum_{L \subset [l]} a_L f_L | \frac{a_{L_1} - a_{L_2}}{v_{k-1}^{L_1}(x_{-\alpha_{i_k}})} \in S,\ \forall L_1, L_2 \text{ such that } L_1 = L_2 \sqcup \{k\}\}.$$

The Bott-Salmelson varieties $\hat{X}_I$ are GKM spaces if $I = (i_1, ..., i_l)$ with i_j all distinct.

Bott-Samelson varieties

Lemma (Calmes-Zainoulline-Zhong)

For any sequence $I = (i_1, \cdots, i_n)$*, we have*
$\imath^*(\zeta_I) = A_{I^{rev}}(pt_e) = A_{\alpha_{i_n}} \cdots A_{\alpha_{i_1}}(pt_e)$*, where*

A_α *is the algebraic realization of* $h_T(G/B) \xrightarrow{\pi_*} h_T(G/P_\alpha) \xrightarrow{\pi^*} h_T(G/B)$.

pt_e *is the image of* $h_T(e/B) \xrightarrow{(\imath_e)_*} h_T(G/B) \xrightarrow{(\imath_e)^*} h_T(e/B)$

Lemma

Let I *be a sequence of length* l *and* $1 \leq k \leq l$*. Denote by* I_k *the subsequence of* I *obtained by removing the* k*-th term from* I*. Then* $\imath^*((q_I)_*(\eta_k)) = A_{I_k^{rev}}(pt_e)$.

Bott-Samelson varieties

Theorem (L.-Zhong)

For any sequence $I = (i_1, \dots, i_l)$, *we have*

$$\imath^* q_{I*}(\eta_L) = \sum_{L_1 \subset L^c} \frac{a_{L,L_1} \cdot v^{L_1}(x_\Pi)}{x_{I,L_1}} f_{v^{L_1}}, \quad x_\Pi := \prod_{\alpha<0} x_\alpha \in S,$$

where $v^L := v_I^L = \prod_{k \in L} s_{i_k}$, *and* $x_{I,L} = \prod_{1 \le j \le l} v_j^L(x_{-i_j})$. *Note that a priori the coefficients of* $f_{v^{L_1}}$ *belong to* S.

Corollary

Let I *be any sequence of length* l. *For any* $L \subset [l]$, *denote by* $q_L : \hat{X}_L \to G/B$. *Then* $q_{I*}(\eta_L) = q_{L^c*}(1)$.

Bott-Samelson varieties

Corollary

For any $u \in h_T(pt)$, *we have*

$$\mathbf{c}'(u) \cdot \zeta_w = \sum_{L \subset [\ell(w)]} \theta_{I,L}(u) \zeta_{L^c},$$

where $\zeta_{L^c} = q_{L^c *}(1)$, $\theta_{I,L} = \theta_1 \cdots \theta_l$ *with*

$$\theta_j = \begin{cases} \Delta_{-\alpha_{i_j}} = X_{-\alpha_{i_j}} \cdot, & \text{if } j \in L, \\ s_{i_j}, & \text{otherwise,} \end{cases} \quad \text{and } \mathbf{c}'(x_\lambda) = c_1(\mathcal{L}_\lambda).$$

Reference

- Calmes, Baptiste, Kirill Zainoulline, and Changlong Zhong. Equivariant oriented cohomology of flag varieties. Doc. Math., Extra Volume: Alexander S. Merkurjev's Sixtieth Birthday (2015), 113-144.
- Calmes, Baptiste, Viktor Petrov, and Kirill Zainoulline. İnvariants, torsion indices and oriented cohomology of complete flags.Ännales Scientifiques de l'Ecole Normale Superieure. Vol. 46. No. 3. 2013.
- Willems, Matthieu. "Cohomologie équivariante des tours de Bott et calcul de Schubert équivariant."Journal of the Institute of Mathematics of Jussieu 5.1 (2006): 125.
- Li, Hao, and Changlong Zhong. " On equivariant oriented cohomology of Bott-Samelson varieties." arXiv preprint arXiv:2004.07680 (2020).

Thank you