Isotopy Graphs for Latin tableaux

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Latin tableaux

3	4	2	1
4	3	1	2
1	2	3	
2	1		

- Young tableau of shape λ, with integer entries.
- Row i contains the numbers $1, 2, 3, \ldots, \lambda_i$.
- No number appears twice in the same column.

A counter-example

- Not every Young diagram can be filled to make a Latin tableau.
- Example: $\lambda =$

- We “get stuck” after filling the bottom two rows:

1	2	
2	1	

Wide partitions

- Compare:

1	2	
2	1	

1	3	2
3	2	1
2	1	

- General criterion: if there is a Latin tableau of shape λ, then λ must be a *wide partition* [Chow et al., 2002].

Rota's Basis Conjecture

Conjecture (Published in [Huang and Rota, 1994])

Suppose we have n bases $\beta_1, \beta_2, \ldots, \beta_n$ of a rank-n matroid. Then we can assign the vectors from these bases to the squares of an $n \times n$ array in such a way that row i contains the elements of β_i; and the n entries assigned to each column of the array also form a basis.

- Question: can we prove Rota's Basic Conjecture by induction on number of boxes in a Young diagram? [Chow et al., 2002]

Rota's Conjecture for Young Diagrams

Definition ([Chow et al., 2002])

Let λ be a partition. We say that λ satisfies Rotas conjecture if, for any matroid M and any sequence I_i of independent sets of M satisfying $|I_i| = \lambda_i$ for all i, there exists an M-tableau T of shape λ such that

(a) *For all i, the set of elements in the i^{th} row of T is I_i.*

(b) *For all j, the elements in the j^{th} column of T comprise an independent set of M.*

Conjecture (The Wide Partition Conjecture, [Chow et al., 2002])

A partition λ satisfies Rota's conjecture if and only if it is wide.

From Rota's Basis Conjecture to Latin tableaux

"Unfortunately, the WPC does not seem to be any easier than Rotas basis conjecture. Nevertheless, we believe that the WPC is interesting in its own right..." [Chow et al., 2002, Page 8].

Conjecture (The WPC for Free Matroids, [Chow et al., 2002])

There exists a Latin tableau of shape λ if and only if λ is wide.

- Chow, Fan, Goeman's and Vondrak proved several special cases of the WPC for free matroids, encouraged others to study Latin tableaux.

Isotopies of Latin tableau

- Latin tableau T and T' are *isotopic* if T' can be obtained from T by some sequence of the following transformations...
  - Switching two rows *of the same length*.
  - Switching two columns *of the same length*.
  - Switching two entries *which appear the same number of times*.
- Example: $T' = \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$ isotopic to $T = \begin{array}{|c|c|c|}\hline 1 & 3 & 2 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$
  - Switch columns 1 and 2, then switch entries 1 and 2.

Isotopy graphs of Latin tableaux

- **Vertices:** Latin tableaux of shape λ.
- **Edges:** T and T' are adjacent if T' is obtained from T by switching two rows, columns or entries.

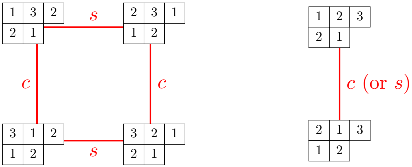

Figure: An isotopy graph.

Investigating isotopy graphs

- Question: What do isotopy graphs of Latin tableaux "look like"?
  - What "nice" graphs appears as subgraphs of isotopy graphs? As connected components?
  - Can we obtain complete graphs? Cubes of dimension d?

Clique numbers of isotopy graphs

Theorem (Roldán, K.)

The isotopy graph of any Latin tableau has clique number 1, 2 *or* 4.

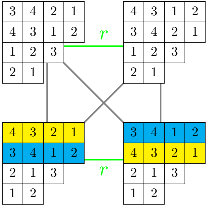

Figure: r switches rows 1 and 2.

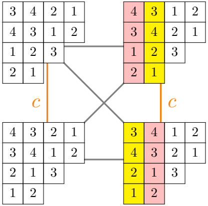

Figure: c switches columns 1 and 2.

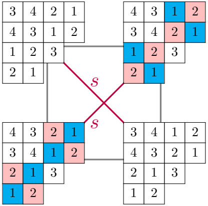

Figure: s switches entries 1 and 2.

Figure: An isotopy graph isomorphic to K_4.

Conventions for d-dimensional Cubes

- Vertices are in bijection with d-tuples of 0's and 1's.
- Two vertices are adjacent if corresponding tuples differ in exactly one place.

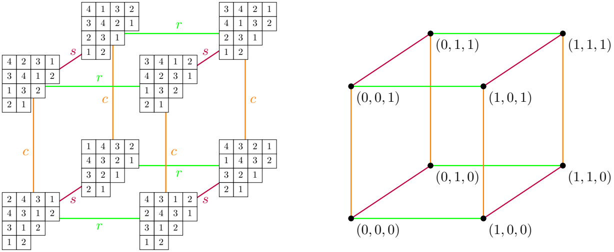

Figure: An isotopy graph isomorphic to a 3-cube.

Cubes in all dimensions

Theorem (Roldán, K.)

A Latin tableau T a Latin tableau of shape λ has an isotopy graph which is a cube if and only if both of the following hold:

1. *The shape λ has no more than two rows or columns of the same length.*
2. *No "nontrivial" isotopy fixes the (filled) tableau T.*

Theorem (Roldán, K.)

For every positive integer d, there exists a Latin tableau T such that the isotopy graph of T is a d-dimensional cube.

Examples: $d = 8, 9$

7	5	3	1	2	4	6	8
5	3	1	2	4	6		
3	1	2	4				
1	2						

Figure: The isotopy graph for this tableau is an 8-dimensional cube.

8	7	5	3	1	2	4	6
7	5	3	1	2	4	6	8
5	3	1	2	4	6		
3	1	2	4				
1	2						

Figure: The isotopy graph for this tableau is a 9-dimensional cube.

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