Degeneracy Loci & Schubert Polynomials

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Old Type A Given vector bundles (or vs with weights) and maps

$$\overset{y_1}{F_1} \subset \overset{y_2}{F_2} \subset \cdots \subset F_q \subset \cdots \to \twoheadrightarrow \cdots \twoheadrightarrow \overset{x_2}{E_p} \twoheadrightarrow \cdots \twoheadrightarrow \overset{x_1}{E_1}$$

rks subscripts. w permutation

$$\Omega_w : rk(F_q \to E_p) \leq \#\{a \leq p \mid w(a) \leq q\}$$

$$[\Omega_w] = \mathfrak{S}_w(x,y) \text{ in } \mathbb{Z}[x,y] \quad \text{(usual meaning)}$$

Type B V vb of odd rank with quad form, $E_\bullet$, $F_\bullet$ complete flags of subbundles. [BH] [IMN]

$$\cdots \subset E_2 \subset E_1 \subset E_0 = E_1^\perp \subset E_{-1} \subset \cdots \subset V \quad (F_\bullet \text{ sim})$$

$rk\, E_p = rk\, E_0 - p$ $\quad w$ signed permutation

$$\Omega_w : \dim E_p \cap F_q \geq \#\{a \leq p \mid -w(a) \geq q\}$$

$$[\Omega_w] = \mathfrak{S}_w^B \in \Gamma[x,y]$$

Γ = poly's in p_i's, basis $Pf_\lambda(p)$

Uses $c_k(V - E_0 - F_1)$ and x's + y's.

Richer! Why? More stable: ranks of bundles not specified!

New Type A

$$\cdots \subset F_q \subset \cdots \subset F_0 \subset \cdots \subset V \to U \twoheadrightarrow \cdots \twoheadrightarrow E_0 \twoheadrightarrow \cdots \twoheadrightarrow E_p \twoheadrightarrow \cdots$$

$$\operatorname{rk} F_0 = \operatorname{rk} E_0$$

$$\operatorname{rk} F_q = \operatorname{rk} F_0 - q \qquad \operatorname{rk} E_p = \operatorname{rk} E_0 - p$$

w permutation (in $S_{\mathbb{Z}}$)

Ω_w: $\dim \operatorname{Ker} F_q \to E_p \geq \#\{a \leq p \mid w(a) > q\}$

Thm $\exists!\ S_w(c,x,y) \in \Lambda[x,y]$, $\Lambda = \mathbb{Z}[c_1, c_2, \ldots]$

so $[\Omega_w] = S_w(c,x,y)$ by

$$c_k \mapsto c_k(E_0 - F_0)$$

$$x_p \mapsto -c_1(\ker(E_{p-1} \to E_p))$$

$$y_q \mapsto c_1(F_{q-1}/F_q)$$

pf Buch–F 1990's!

Properties (all of them)

$S_w(0,x,y) = \mathfrak{S}_w(x,-y)$ $\quad w \in S_\infty$

$S_{w^{-1}}(c,x,y) = \mathfrak{S}_w(c^*,y,x)$ $\quad$ (take duals)

$S_{\gamma^m w}(c,x,y) = \gamma^m \mathfrak{S}_w(c,x,y)$ $\quad$ back-stable

$\gamma^m w(m+i) = m + w(i)$ $\quad (\gamma^m x)_i = x_{m+i}, \ldots$

$\gamma^m c = c \cdot \prod_{i=1}^{m} \frac{1+y_i t}{1-x_i t}$ $\quad c = \sum c_i t^i$

In fact $S_w \mapsto \overleftarrow{\mathfrak{S}}_w$ of LLS (cf B & Kn)

- Difference ops
- $S_w(c,0,0) = F_w$ Stanley

$S_u S_v = \sum c_{uv}^w S_w \qquad c_{uv}^w \in \mathbb{Z}_{\geq 0}[y_i - y_j]$ (Graham)

Det formulas for w vexillary (2143-avoiding)

or 321-avoiding

These are irreducible

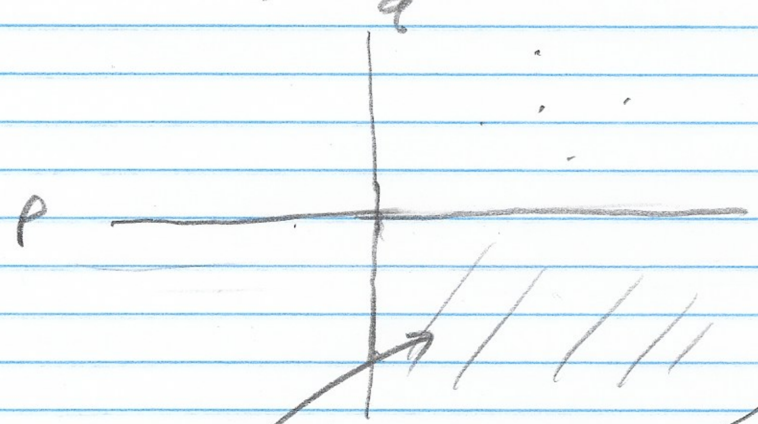
both $\mathrm{Det}(c(k,\ell)_{\lambda_k + \ell - k})$

Localization formulas

Type A & C

New Matrix Sch Vars $\qquad A = (a_{ij})$

$\mathrm{wt}\, a_{ij} = -x_i - y_j \qquad w$ perm.



Nullity $\leq$ $\#\{a \leq p \mid w(a) \geq q\}$

Nullity of $m \times n$ is dim ker $K^n \to K^m$