A Pieri rule for the quantum K-theory of OG(n,2n)

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$X = G/P_X$ flag variety. $T \subset B \subset P_X \subset G$

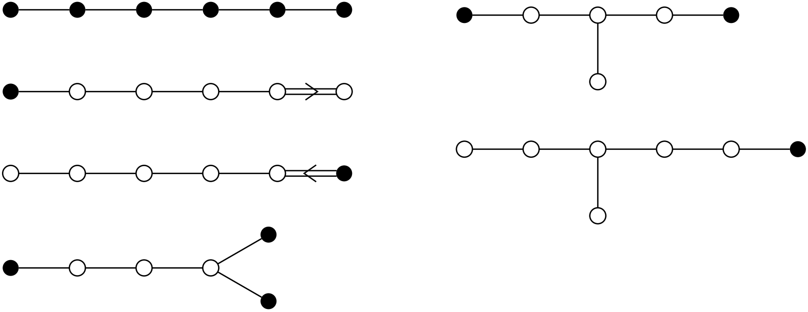
$W = N_G(T)/T$ Weyl group. $W_X = N_{P_X}(T)/T$ Weyl group of P_X.
$W^X \subset W$ minimal representatives of cosets in W/W_X.

Schubert varieties: For $w \in W$ set $X_w = \overline{Bw.P_X}$ and $X^w = \overline{B^- w.P_X}$
$w \in W^X \Rightarrow \dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$

K-theory ring: $K(X) = \bigoplus_{w \in W^X} \mathbb{Z}\,[\mathcal{O}_{X^w}]$

$[\mathcal{O}_{X^u}] \cdot [\mathcal{O}_{X^v}] = \sum_w C_{u,v}^w [\mathcal{O}_{X^w}]$ Brion: $(-1)^{\ell(wuv)} C_{u,v}^w \geq 0$

Cominuscule simple roots

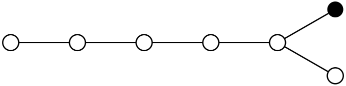


Cominuscule flag variety: G/P_α with α is cominuscule.

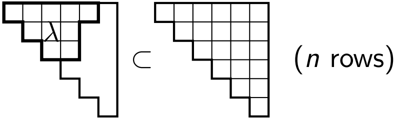
Minuscule flag variety: G/P_α with α is cominuscule, G simply laced.

Maximal orthogonal Grassmannians

$X = \mathrm{OG}(n+1, 2n+2) =$ component in $\{V \subset \mathbb{C}^{2n+2} \mid V \text{ max isotropic}\}$



$W^X \longleftrightarrow$ strict partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$



$\mathrm{codim}(X^\lambda, X) = |\lambda| = \#$ boxes in λ

Pieri formula for $K(X)$, $X = \mathrm{OG}(n+1, 2n+2)$

Let $\lambda \subset \nu$ be strict partitions.

A **KOG tableau** of shape ν/λ is a labeling of the boxes in ν/λ with integers such that

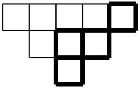
(1) All rows and columns are strictly increasing, and

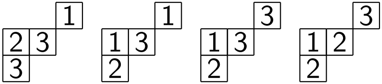
(2) Each label is either $\leq$ all labels south-west of it, or $\geq$ all labels south-west of it.

Theorem (B-Ravikumar) $[\mathcal{O}_{X^{(p)}}] \cdot [\mathcal{O}_{X^\lambda}] \;=\; \sum_\nu C^\nu_{p,\lambda}\, [\mathcal{O}_{X^\nu}]$ in $K(X)$

$C^\nu_{p,\lambda} = (-1)^{|\nu/\lambda| - p}$ # KOG-tableau of shape ν/λ with content $\{1, \ldots, p\}$

Example: $\nu = (5,3,1)$, $\lambda = (4,1)$, $p = 3$. Then $C^\nu_{3,\lambda} = -4$.

$\nu/\lambda =$ 



Gromov-Witten invariants

$M_d = \overline{\mathcal{M}}_{0,3}(X,d) = \overline{\{f : \mathbb{P}^1 \to X \text{ of degree } d\}}$ Kontsevich moduli space.

Let $\Omega_1, \Omega_2, \Omega_3 \subset X$ be closed, in general position.

Gromov-Witten variety:

$M_d(\Omega_1, \Omega_2, \Omega_3) = \{f \in M_d \mid f(0) \in \Omega_1, f(1) \in \Omega_2, f(\infty) \in \Omega_3\}$

$$\langle [\Omega_1], [\Omega_2], [\Omega_3] \rangle_d \;=\; \begin{cases} \# M_d(\Omega_1, \Omega_2, \Omega_3) & \text{if finite} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \langle [\mathcal{O}_{\Omega_1}], [\mathcal{O}_{\Omega_2}], [\mathcal{O}_{\Omega_3}] \rangle_d \;&=\; \chi(\mathcal{O}_{M_d(\Omega_1,\Omega_2,\Omega_3)}) \\ &=\; \sum_{p \geq 0} (-1)^p \dim H^p(M_d(\Omega_1, \Omega_2, \Omega_3), \mathcal{O}) \end{aligned}$$

Quantum cohomology

$$QH(X) = H(X) \otimes \mathbb{Z}[q]$$

$$[X^u] \star [X^v] \;=\; \sum_{w,d\geq 0} \langle [X^u], [X^v], [X_w] \rangle_d \, q^d \, [X^w]$$

Quantum K-theory

$$\mathrm{QK}(X) = K(X) \otimes \mathbb{Z}[[q]]$$

$$[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}] = \sum_{w,d\geq 0} N_{u,v}^{w,d} \, q^d \, [\mathcal{O}_{X^w}]$$

where $N_{u,v}^{w,d} \;=\; \langle \mathcal{O}_{X^u}, \mathcal{O}_{X^v}, \mathcal{I}_w \rangle_d \;-\; \sum_{\kappa, 0<e\leq d} N_{u,v}^{\kappa, d-e} \, \langle \mathcal{O}^\kappa, \mathcal{I}_w \rangle_d$

and $\mathcal{I}_w \subset \mathcal{O}_{X_w}$ ideal sheaf of $\partial X_w = \bigcup_{w'<w} X_{w'}$

Results about QK(X)

Finiteness: $N_{u,v}^{w,d} = 0$ for large degrees d.

Gr(m,n) [BM], Pic(X)=$\mathbb{Z}$ [BCMP],

G/B [Kato], G/P [Anderson-Chen-Tseng]

Functoriality:

$\exists$ ring hom. $\mathrm{QK}(G/P) \to \mathbb{Z}$ [B-Chung], [B-Chung-Li-Mihalcea]

$\exists$ ring hom. $\mathrm{QK}(G/P) \to \mathrm{QK}(G/Q)$ whenever $P \subset Q$ [Kato]

Chevalley formula for product with divisor in QK(X):

X cominuscule [BCMP]

$X = G/B$ [Lenart-Naito-Sagaki]

Positivity conjecture:

$(-1)^{\ell(uvw)+\deg(q^d)} N_{u,v}^{w,d} \geq 0$ where $\deg(q^d) = \int_d c_1(T_X)$

Proved if X is minuscule or quadric hypersurface [BCMP]

Powers of q in quantum products

Theorem (Postnikov)

$X = \mathrm{Gr}(m, n) \Rightarrow$ powers of q in $[X^u] \star [X^v] \in QH(X)$ is an interval.

Theorem (BCMP)

X cominuscule $\Rightarrow$ powers of q in $[X^u] \star [X^v] \in QH(X)$ is an interval.

X cominuscule $\Rightarrow$ powers of q in $[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}] \in \mathrm{QK}(X)$ is an interval.

X minuscule $\Rightarrow$ $[X^u] \star [X^v]$ and $[\mathcal{O}_{X^u}] \star [\mathcal{O}_{X^v}]$ contain **same** powers of q.

Example:

$X = \mathrm{Fl}(6)$, $w = 164532$.

Powers of q in $[X^w]^2 \in QH(X)$ do not form an interval.

Seidel representation

Given $Y = G/P_Y$, let $\pi_Y \in W^Y$ be the longest element ($Y^{\pi_Y} = \text{point}$).

Theorem (Chaput-Manivel-Perrin):

Y cominuscule, X any flag variety $\Rightarrow$

$[X^{\pi_Y}] \star [X^w] = q^{d(Y,w)}\,[X^{\pi_Y w}]$ in $QH(X)$

Theorem (BCMP):

Y and X both cominuscule $\Rightarrow$

$[\mathcal{O}_{X^{\pi_Y}}] \star [\mathcal{O}_{X^w}] = q^{d(Y,w)}\,[\mathcal{O}_{X^{\pi_Y w}}]$ in $\mathrm{QK}(X)$

Example: $X = \mathrm{OG}(n+1, 2n+2)$, $Y = \mathrm{OG}(1, 2n+2)$

$X^{\pi_Y} = X^{(n)} = X^{\square\square\square\square\square\square}$

$$[\mathcal{O}_{X^{(n)}}] \star [\mathcal{O}_{X^\lambda}] = \begin{cases} [\mathcal{O}_{X^{(n,\lambda)}}] & \text{if } \lambda_1 < n \\ q[\mathcal{O}_{X^{\overline{\lambda}}}] & \text{if } \lambda_1 = n, \quad \text{where } \overline{\lambda} = (\lambda_2, \ldots, \lambda_\ell) \end{cases}$$

Pieri formula for $\mathrm{QK}(X)$, $X = \mathrm{OG}(n+1, 2n+2)$

Compute $[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}]$ in $\mathrm{QK}(X)$

Assume $\lambda_1 < n$:

$[X^{(p)}] \star [X^\lambda]$ has no q-terms $\Rightarrow$ $[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}]$ has no q-terms.

$$[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}] \;=\; [\mathcal{O}_{X^{(p)}}] \cdot [\mathcal{O}_{X^\lambda}] \;=\; \sum_\nu C^\nu_{p,\lambda} \, [\mathcal{O}_{X^\nu}]$$

Assume $\lambda_1 = n$:

$$[\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^\lambda}] \;=\; [\mathcal{O}_{X^{(p)}}] \star [\mathcal{O}_{X^{\overline{\lambda}}}] \star [\mathcal{O}_{X^{(n)}}] \;=\; \sum_\nu C^\nu_{p,\overline{\lambda}} \, [\mathcal{O}_{X^\nu}] \star [\mathcal{O}_{X^{(n)}}]$$