PLAQUETTE PERCOLATION ON THE TORUS

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Abstract. We study plaquette percolation on a d-dimensional torus $\mathbb{T}_N^d$, defined by identifying opposite faces of the cube $[0,N]^d$. The model we study starts with the complete $(i-1)$-dimensional skeleton of the cubical complex $\mathbb{T}_N^d$ and adds i-dimensional cubical plaquettes independently with probability p. Our main result is that if $d = 2i$ is even and $\phi_* : H_i(P;\mathbb{Q}) \to H_i(\mathbb{T}^d;\mathbb{Q})$ is the map on homology induced by the inclusion $\phi : P \hookrightarrow \mathbb{T}^d$, then

$$\mathbb{P}_p(\phi_* \text{ is nontrivial}) \to \begin{cases} 0 & \text{if } p < \frac{1}{2} \\ 1 & \text{if } p > \frac{1}{2} \end{cases}$$

as $N \to \infty$. We also show that 1-dimensional and $(d-1)$-dimensional plaquette percolation on the torus exhibit similar sharp thresholds at $\hat{p}_c$ and $1 - \hat{p}_c$ respectively, where $\hat{p}_c$ is the critical threshold for bond percolation on $\mathbb{Z}^d$, as well as bounds on critical probabilities in other dimensions.

1. Introduction

Various models of percolation are fundamental in statistical mechanics; classically, they study the emergence of a giant component in random structures. From early in the mathematical study of percolation, geometry and topology have been at the heart of the subject. Indeed, Frisch and Hammersley wrote in 1963 [8] that, “Nearly all extant percolation theory deals with regular interconnecting structures, for lack of knowledge of how to define randomly irregular structures. Adventurous readers may care to rectify this deficiency by pioneering branches of mathematics that might be called *stochastic geometry* or *statistical topology*.”

The Harris–Kesten theorem [5, 11, 13] establishes that for bond percolation in the square lattice, the critical probability for an infinite component to appear is $p = 1/2$. A key topological lemma in proving it concerns duality for crossings

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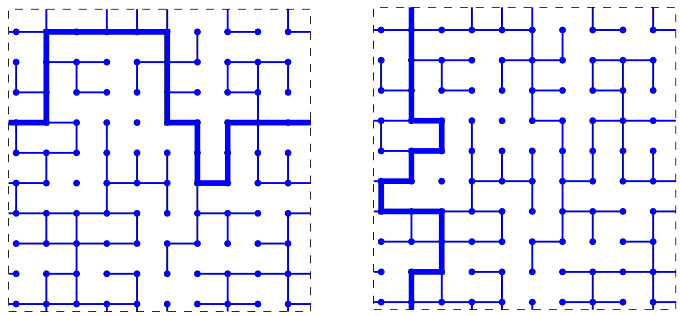

FIGURE 1. Two giant cycles for a random system of 1-dimensional plaquettes (bonds) on a 2-dimensional torus, shown in a square with opposite sides identified.

of a rectangle: for any rectangle R, there is a horizontal crossing of a rectangle via edges in the bond system if and only if there is no vertical crossing in the dual bond system. This suggests the potential utility of duality theorems from algebraic topology for the purpose of establishing analogous results in higher dimensions.

The topological study of percolation with higher-dimensional cells was initiated by Aizenman, Chayes, Chayes, Fróhlich, and Russo in [1], with a 2-dimensional "plaquette" model in $\mathbb{Z}^3$. This model starts with the entire 1-skeleton of $\mathbb{Z}^3$ and adds 2-dimensional square cells, or plaquettes, with probability p independently. They prove that the probability that a large planar loop is null-homologous undergoes a phase transition from an "area law" to a "perimeter law" that is dual to the phase transition for bond percolation in $\mathbb{Z}^3$. In particular, the critical probability for this threshold is at $p_c = 1 - \hat{p}_c$, where $\hat{p}_c$ is the threshold for bond percolation (this follows when their results are combined with [10]). At the end of their paper, the authors suggested the study of analogous questions in higher dimensions.

> Of particular interest are random $d/2$-cells in even dimension d. Clearly, if no intermediate phase exists in such a self-dual system, the transition point is $p = 1/2$. The most promising model for future study is random plaquettes in $d = 4$. [1]

Inspired by [1], we study the following *i-dimensional plaquette percolation model P*. Let $N \geq 1$ and let $T_N^d = \mathbb{Z}^d/(N\mathbb{Z})^d$ be the d-dimensional torus. Take the entire $(i-1)$-skeleton of T_N^d and add each i-face independently with probability p.

The *homological percolation property* we consider is one that was recently introduced by Bobrowski and Skraba [2]. For a random shape S on the torus

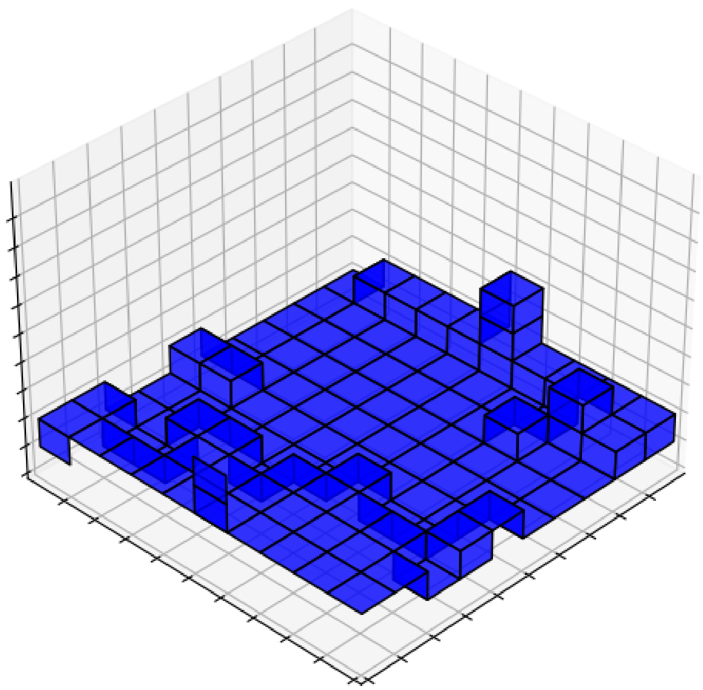

FIGURE 2. A giant cycle for 2-dimensional plaquette percolation on a 3-dimensional torus, shown in a cube with opposite sides identified.

$\mathbb{T}^d$ (they study several different models), let ϕ denote the inclusion $S \hookrightarrow \mathbb{T}^d$. They suggest that nontrivial elements in the image of the induced map on i-dimensional homology $\phi_* : H_i(P; \mathbb{Q}) \to H_i\left(\mathbb{T}^d; \mathbb{Q}\right)$ can be considered "giant i-dimensional cycles". In the case $i = 1$, these correspond to paths that loop around the torus as in Figure 1 whereas for the case $i = 2$ and $d = 3$ they are sheets going from one side of the torus to the other, as illustrated in Figure 2. In this first paper, they provide experimental evidence that the appearance of giant cycles is closely related to the zeroes of the expected Euler characteristic curve, and moreover that this behavior seems universal across several percolation models.

Our present work is more closely related to Bobrowski and Skraba's more recent paper, [3]. In this paper, they study this notion of homological percolation in the continuum percolation (or random geometric complex) model, showing that giant cycles appear in all dimensions within the so-called thermodynamic limit where nr^d is bounded, and they describe a sharp phase transition for one-dimensional percolation.

We use similar topological tools, including a duality lemma that is similar to Bobrowski and Skraba's. We also find similar inequalities between critical probabilities for giant cycles of different dimensions.

There are important differences between the continuum and plaquette models however, and we wish to contrast those as well. The discrete nature of the plaquettes lends itself to more combinatorial arguments, and has some nice properties with respect to duality as well. In particular, we show that the complement of the plaquette process deformation retracts to a dual plaquette process. This leads to relationships between plaquettes of complementary dimensions and, in particular, self-duality in the middle dimension.

1.1. **Main results.** Let $d, N \in \mathbb{N}$, $d > 1$, $1 \leq i \leq d-1$ and $p \in (0,1)$. As above, denote by $T_N^d = \mathbb{Z}^d/(N\mathbb{Z})^d$ the regular cubical complex on the d-dimensional torus with N^d cubes of width one. Let $H_i(X)$ be homology with coefficients in $\mathbb{Q}$.

Define the plaquette system $P = P(i, d, N, p)$ to be the random set obtained by taking the $(i-1)$-skeleton of $\mathbb{T}_N^d$ and adding each i-face independently with probability p. Let $\phi : P \hookrightarrow \mathbb{T}^d$ be the inclusion, and let $\phi_* : H_i(P) \to H_i\left(\mathbb{T}_N^d\right)$ be the induced map on homology. Also, denote by $A = A(i, d, N, p)$ the event that ϕ_* is nontrivial, and denote by $S = S(i, d, N.p)$ the event that ϕ_* is surjective. For example, in Figure 1 the two giant cycles shown are homologous with standard generators for $H_1(T^2)$, so we have the event S.

Our main result is that if $d = 2i$, then i-dimensional percolation is self-dual and undergoes a sharp transition at $p = 1/2$.

Theorem 1. *If* $d = 2i$ *then*

$$\begin{cases} \mathbb{P}_p(A) \to 0 & p < \frac{1}{2} \\ \mathbb{P}_p(S) \to 1 & p > \frac{1}{2} \end{cases}$$

as $N \to \infty$.

Using results on bond percolation on $\mathbb{Z}^d$, we also prove dual sharp thresholds for 1-dimensional and $(d-1)$-dimensional percolation on the torus.

Theorem 2. *Let* $\hat{p}_c = \hat{p}_c(d)$ *be the critical threshold for bond percolation on* $\mathbb{Z}^d$. *If* $i = 1$ *then*

$$\begin{cases} \mathbb{P}_p(A) \to 0 & p < \hat{p}_c \\ \mathbb{P}_p(S) \to 1 & p > \hat{p}_c \end{cases}$$

as $N \to \infty$.

Furthermore, if $i = d-1$ then

$$\begin{cases} \mathbb{P}_p(A) \to 0 & p < 1 - \hat{p}_c \\ \mathbb{P}_p(S) \to 1 & p > 1 - \hat{p}_c \end{cases}$$

as $N \to \infty$.

In the above, we also show that the decay of $\mathbb{P}_p(A)$ below the threshold and $\mathbb{P}_p(S)$ above the threshold is exponentially fast for both $i = 1$ and $i = d - 1$.

For other values of i, d we show the following. Define

$$p_c(i,d) = \inf\left\{p : \liminf_{N\to\infty} \mathbb{P}_p(A) > 0\right\}$$

and

$$q_c(i,d) = \sup\left\{p : \limsup_{N\to\infty} \mathbb{P}_p(S) < 1\right\}.$$

With the understanding that these depend on choice of i and d, which are always understood in context, we sometimes abbreviate to simply p_c and q_c.

Theorem 3. *For every $d \geq 2$ and $1 \leq i \leq d-1$, we have $0 < q_c \leq p_c < 1$ and*

$$\begin{cases} \mathbb{P}_p(A) \to 0 & p < q_c \\ \mathbb{P}_p(S) \to 1 & p > p_c \end{cases}$$

as $N \to \infty$.

Moreover, $p_c(i,d)$ has the following properties.

(a) (Duality) $p_c(i,d) + q_c(d-i,d) = 1$.

(b) (Monotonicity in i and d) $p_c(i,d) < p_c(i,d-1) < p_c(i+1,d)$ if $0 < i < d-1$.

It follows that $p_c = q_c$ for $i = d/2$, $i = 1$, and $i = d-1$, and we conjecture that this equality (and hence sharp threshold from a trivial map to a surjective one) holds for all i and d. Bobrowksi and Skraba make analogous conjecture for the continuum percolation model in [3].

1.2. **Probabilistic tools.** Let X be the probability space formed by the product of n Bernoulli(p) random variables, and let μ_p be the probability measure on the power set $\mathcal{P}(X)$ defined by taking each element of X independently with probability p. That is, if $Y \subseteq X$,

$$\mu(Y) = p^{|Y|}(1-p)^{|Y|} .$$

An event B is **increasing** if

$$Y_0 \subset Y_1, Y_0 \in B \implies Y_1 \in B .$$

We require Harris's Inequality on increasing events [11], which is a special case of the FKG Inequality [6].

Theorem 4 (Harris's Inequality)**.** *If $B_1, \ldots, B_j$ are increasing events then*

$$\mathbb{P}\left(\bigcap_{k=1}^{j} B_k\right) \geq \prod_{k=1}^{j} \mathbb{P}(B_k) .$$

Another key tool for us is the following theorem of Friedgut and Kalai on sharpness of thresholds [7].

Theorem 5 (Friedgut and Kalai)**.** *Let B be an increasing event that is invariant under a transitive group action on X. There exists a constant $\rho > 0$ so that if $\mu_p(B) > \epsilon > 0$ and*

$$q \geq p + \rho \frac{\log(1/(2\epsilon))}{\log(|X|)} \tag{1}$$

then $\mu_q(B) > 1 - \epsilon$.

We use two more technical results on connection probabilities in the subcritical and supercritical phases in bond percolation in $\mathbb{Z}^d$ below. For clarity, we state them in Section 6 when they are needed.

1.3. **Definitions and notation.** To define the *dual system of plaquettes* $P^\bullet = P^\bullet(i,d,N,p)$, let $(\mathbb{T}_N^d)^\bullet$ be the regular cubical complex obtained by shifting $\mathbb{T}_N^d$ by $\frac{1}{2}$ in each coordinate direction. Each i-face of $\mathbb{T}_N^d$ intersects a unique $(d-i)$-face of $(\mathbb{T}_N^d)^\bullet$, and they meet in a single point at their centers. For example, the faces $[0,1]^i \times \{0\}^{d-i}$ and $\{1/2\}^i \times [-1/2,1/2]^{d-i}$ intersect in the point $\left\{\frac{1}{2}\right\}^i \times \{0\}^{d-i}$.

Define the dual system $P^\bullet$ to be the subcomplex of $(\mathbb{T}_N^d)^\bullet$ consisting of all faces for which the corresponding face in $\mathbb{T}_N^d$ is not contained in P. See Figure 1.3. Observe that the distribution of $P^\bullet(i,d,N,p)$ is the same as that of $P(d-i,d,N,1-p)$. If $B^\bullet$ is an event defined in terms of $P^\bullet$ we will write

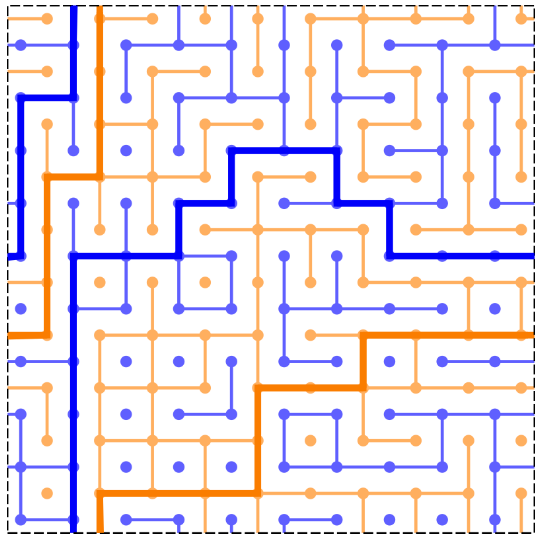

FIGURE 3. Bond percolation at criticality (i.e. $p = 1/2$) on the torus T^2_{10} in blue, with the corresponding dual system of bonds in orange. Giant cycles are shown in bold. Observe that while $\operatorname{rank}\phi_* + \operatorname{rank}\psi_* = 2$ (as required by duality), neither the bond system nor its dual has a giant cycle homologous to one of the standard basis elements of $H_1(\mathbb{T}^2)$.

$\mathbb{P}_p(B^\bullet)$ to mean the probability of $B^\bullet$ with respect to the parameter p of P.

We always use the notation $\phi : P \hookrightarrow \mathbb{T}^d$ and $\psi : P^\bullet \hookrightarrow \mathbb{T}^d$ for the respective inclusion maps, and $\phi_* : H_i(P) \to H_i\left(\mathbb{T}^d_N\right)$ and $\psi_* : H_{d-i}(P^\bullet) \to H_{d-i}\left(\mathbb{T}^d\right)$ for the induced maps on homology. Also, we consistently use notation $A = A(i, d, N, p)$ for the event that $\operatorname{im}\phi_* \neq 0$, $S = S(i, d, N, p)$ for the event that ϕ_* is surjective, and $Z = Z(i, d, N, p)$ for the event that ϕ_* is zero. Denote by $A^\bullet, S^\bullet$, and $Z^\bullet$ the corresponding events for ψ_*.

Throughout this paper, we always write $H_i(X)$ for the i-dimensional homology of a topological space X with rational coefficients.

1.4. **Sketch of proof.** We provide an overview of our main argument. In the section on topological results (Section 2), we show that duality holds in the sense that $\operatorname{rank}\phi_* + \operatorname{rank}\psi_* = \operatorname{rank} H_i\left(\mathbb{T}^d\right)$ (Lemma 8). This is similar in spirit to results of [2] and [3] for other models of percolation on the torus. In particular, at least one of the events A and $A^\bullet$ occurs, S occurs if and only if $Z^\bullet$ occurs, and $S^\bullet$ occurs if and only if Z occurs. In this section, we also discuss the action of the symmetry group of $\mathbb{T}^d$ on the homology.

Our strategy is to exploit the duality between the events S and $Z^\bullet = (A^\bullet)^c$. Toward that end, we show that a threshold for A is also a threshold for S in Section 3. First, we use the action of the symmetry group of the torus on the homology to show that there are constants b_0 and b_1 so that $\mathbb{P}_p(S) \geq b_0 \mathbb{P}_p(A)^{b_1}$. This is done in two steps: the action of the reflections yields a bound on the probability that $\operatorname{im}\phi_*$ contains one of the standard basis elements of $H_i\left(\mathbb{T}^d\right)$ (Lemma 9), and applying rotations to that event allows us to bound the probability that $\operatorname{im}\phi_*$ contains a basis (Lemma 10).

Recall that

$$p_c = p_c(i,d) = \inf\left\{p : \liminf_{N\to\infty} \mathbb{P}_p(A) > 0\right\}.$$

Let $p > p_c$ so there is a $\delta > 0$ such that $\mathbb{P}_p(A) > \delta$ for all sufficiently large N. By the above, it follows that $\mathbb{P}_p(S) > b_0\delta^{b_1}$ for all sufficiently large N. S is increasing and invariant under the symmetry group of $\mathbb{T}^d$ so Friedgut and Kalai's theorem on sharpness of thresholds, Theorem 5, implies that $\mathbb{P}_p(S) \to 1$.

The proof of Theorem 1 is then straightforward (Section 4). By duality

$$\mathbb{P}_{1/2}(A) = \mathbb{P}_{1/2}(A^\bullet) \qquad \text{and} \qquad \mathbb{P}_{1/2}(A) + \mathbb{P}_{1/2}(A^\bullet) \geq 1\,,$$

so $\mathbb{P}_{1/2}(A) \geq \frac{1}{2}$ for all N. It follows that $\mathbb{P}_p(S) \to 1$ for $p > 1/2$. On the other hand, if $p < 1/2$ duality implies that

$$\mathbb{P}_p(A) = 1 - \mathbb{P}_p(S^\bullet) = 1 - \mathbb{P}_{1-p}(S) \to 0\,.$$

Next, in Section 5 we study the relationship between duality and sharpness. Recall that

$$q_c(i,d) = \sup\left\{p : \limsup_{N\to\infty} \mathbb{P}_p(S) < 1\right\}.$$

We show that $p_c(i,d) + q_c(d-i,d) = 1$ by using Lemma 8 and applying Theorem 5 to A above $p_c(d-i,d)$ and to $A^\bullet$ below $q_c(i,d)$ (Proposition 14). It follows that A undergoes a sharp phase transition if and only if $p_c(d-i,d) + p_c(i,d) = 1$ (Corollary 15).

In Section 6 we show that $p_c(1,d)$ and $q_c(1,d)$ coincide and equal the critical threshold for bond percolation on $\mathbb{Z}^d$ by applying classical results on connection probabilities in the subcritical and supercritical regimes (in the proofs of Propositions 19 and 17). This together with Corollary 15 demonstrates Theorem 2.

Finally, in Section 7, we complete the proof of Theorem 3 by showing the monotonicity property $p_c(i,d) < p_c(i,d-1) < p_c(i+1,d)$ if $0 < i < d-1$, and corresponding result for the thresholds q_c (Proposition 20). This is done by comparing percolation on $\mathbb{T}_N^d$ with percolation on the slice $\mathbb{T}_N^d \cap \{0 \leq x_1 \leq 1\}$, where x_1 is the first coordinate of the flat torus $\mathbb{T}^d$.

2. Topological Results

In this section, we prove a duality lemma which will be useful in many of our arguments. We also discuss the structure of the homology group $H_i\left(\mathbb{T}^d\right)$.

Lemma 6. *$\mathbb{T}^d \setminus P$ deformation retracts to $P^\bullet$.*

Proof. Let $T^{(j)}$ and $T_\bullet^{(j)}$ denote the j-skeletons of $\mathbb{T}_N^d$ and $\left(\mathbb{T}_N^d\right)^\bullet$, respectively, and let

$$S_j = T_\bullet^{(d-j)} \setminus T^{(j)}\,.$$

Observe that S_j is obtained from $T_\bullet^{(d-j)}$ by removing the central point of each $d-j$-cell. Also, let

$$\hat{S}_j = T_\bullet^{(d-j)} \setminus P\,.$$

We construct a deformation retraction from $T^d \setminus P = \hat{S}_0$ to $P^\bullet$ by iteratively collapsing $\hat{S}_j$ to $\hat{S}_{j+1}$ for $j < i$, then collapsing $\hat{S}_i$ to $P^\bullet$.

For an j-cell σ of $T_\bullet^{(j)}$ with center point q let

$$f_\sigma : \sigma \setminus \{q\} \times [0,1] \to \sigma \setminus \{q\}$$

be the deformation retraction from the punctured j-dimensional cube to its boundary along straight lines radiating from the center. Observe that the restriction of f_σ to $(\sigma \setminus P) \times [0,1]$ defines a deformation retraction from $\sigma \setminus P$ to $\partial\sigma \setminus P$ (for $j > d-i$); this is because σ intersects P in hyperplanes spanned by the coordinate vectors based at q. When projecting radially from q, points inside $\sigma \cap P$ remain inside $\sigma \cap P$ and points outside of $\sigma \cap P$ remain outside of $\sigma \cap P$.

For $x \in \mathbb{T}^d$, let $\sigma(x)$ be the unique $d-j$-cell of $(\mathbb{T}_N^d)^\bullet$ that contains x in its interior. Define $G_j : S_j \times [0,1] \to S_j$ by

$$G_j(x,t) = \begin{cases} f_{\sigma(x)}(x,t) & x \in S_j \setminus T_\bullet^{d-j-1} \\ x & \text{otherwise}\,. \end{cases}$$

G_j collapses S_j to $T^{(d-j-1)}$ by retracting the punctured $d-j$-cells to their boundaries. It follows from the discussion in the previous paragraph that the restriction of G_j to $\hat{S}_j \times [0,1]$ defines a deformation retraction from $\hat{S}_j$ to $\hat{S}_{j+1}$.

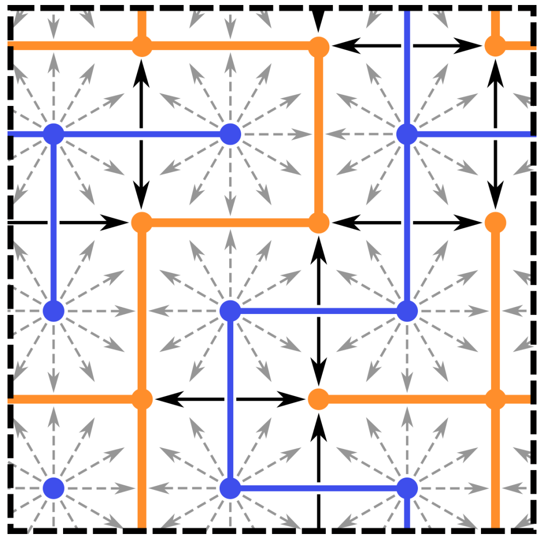

FIGURE 4. of the deformation retraction for the case $N = 3$, $d = 2, i = 1$. P is shown in blue and $P^\bullet$ in orange. $\mathbb{T}^d \setminus P$ is first retracted to $T_\bullet^{(1)} \setminus P$ via the dashed gray arrows radiating from each vertex of P, then to $P^\bullet$ by the solid black arrows radiating from the midpoints of the edges of P.

Similarly, define $H : \hat{S}_i \times [0,1] \to \hat{S}_i$ by

$$H(x,t) = \begin{cases} f_{\sigma(x)}(x,t) & x \in \hat{S}_i \setminus P^\bullet \\ x & \text{otherwise}. \end{cases}$$

That is, H collapses the $(d-i)$-faces of $T_\bullet^{(d-i)}$ that are punctured by i-faces of P to deformation retract $\hat{S}_i$ to $P^\bullet$.

In summary, we can deformation retract $\mathbb{T}^d \setminus P$ to $P^\bullet$ via the function $F : T^d \setminus P \times [0,i] \to T^d \setminus P$ defined by

$$F(x,t) = \begin{cases} G_0(x,t) & t \in [0,1] \\ G_j(F(x,j), t-j) & t \in (j, j+1], 0 < j < i \\ H(F(x,i), t-i) & t \in (i, i+1]. \end{cases}$$

□

In fact, the same deformation retraction works when P is slightly thickened, which will be useful for the next Lemma. Let P_ϵ denote the ϵ-neighborhood

$$P_\epsilon = \{x \in \mathbb{T}_N^d : d(x,P) \leq \epsilon\}.$$

Corollary 7. *For any $0 < \epsilon < 1/2$, the closure $\overline{\left(\mathbb{T}^d \setminus P_\epsilon\right)}$ deformation retracts to $P^\bullet$.*

Proof. Consider the deformation retraction as in Lemma 6 restricted to the closure $\overline{\left(\mathbb{T}^d \setminus P_\epsilon\right)}$. When a punctured j-cell σ is retracted via f_σ, the property that points outside of $\sigma \cap P_\epsilon$ remain outside of $\sigma \cap P_\epsilon$ is preserved even though $\sigma \cap P_\epsilon$ now is a union of thickened hyperplanes. The deformation retractions G_j and H are defined by collapsing different cells via the functions f_σ, so the restricted retraction does not pass through P_ϵ. □

The next result is a key topological tool we use in many of our arguments. It is very similar to results of [2] and [3] for other models of percolation on the torus. For convenience, let

$$D = \operatorname{rank} H_i\left(\mathbb{T}^d\right) = \binom{d}{i}.$$

Lemma 8 (Duality Lemma)**.** $\operatorname{rank}\phi_* + \operatorname{rank}\psi_* = D$. *In particular, at least one of the events A and $A^\bullet$ occurs, $S^\bullet \iff Z$, and $Z \iff S^\bullet$.*

Proof. We proceed similarly to the proof of Lemma C.2 in [2]. Let $\epsilon = 1/4$ and define $P_\epsilon^c \coloneqq \overline{\left(\mathbb{T}_N^d \setminus P_\epsilon\right)}$. Consider the diagram

$$\begin{array}{ccccccc}
H_i\left(P_\epsilon\right) & \xrightarrow{\quad i_* \quad} & H_i\left(\mathbb{T}_N^d\right) & \longrightarrow & H_i\left(\mathbb{T}_N^d, P_\epsilon\right) & \xrightarrow{\quad \delta_i \quad} & H_{i-1}\left(P_\epsilon\right) \\
\uparrow\cong & & \uparrow\cong & & \uparrow\cong & & \uparrow\cong \\
H^{d-i}\left(\mathbb{T}_N^d, P_\epsilon^c\right) & \longrightarrow & H^{d-i}\left(\mathbb{T}_N^d\right) & \xrightarrow{\ j^*\ } & H^{d-i}\left(P_\epsilon^c\right) & \xrightarrow{\ \delta^{d-i}\ } & H^{d-i+1}\left(\mathbb{T}_N^d, P_\epsilon^c\right)
\end{array}$$

Here i and j are the inclusions of P_ϵ and P_ϵ^c respectively into $\mathbb{T}_N^d$. The first isomorphism from the left is from Lefschetz Duality, the second is from Poincaré Duality, and the third is from the five lemma. (A similar diagram is used in the proof of Alexander duality.) In particular note that by exactness and a diagram chase,

$$H_i(\mathbb{T}_N^d) \cong \operatorname{im} i_* \oplus \operatorname{im} j^*.$$

Furthermore, since we are considering homology with field coefficients, $\operatorname{rank} j^* = \operatorname{rank} j_*$. Now by Corollary 7, $\overline{\left(\mathbb{T}_d^N \setminus P_\epsilon\right)}$ retracts to $P^\bullet$, and P_ϵ clearly retracts to P, so $\operatorname{rank}\phi_* = \operatorname{rank} i_*$ and $\operatorname{rank}\psi_* = \operatorname{rank} j^*$. Putting these together gives $\operatorname{rank}\phi_* + \operatorname{rank}\psi_* = D$. □

Before proceeding to probabilistic arguments, we describe the structure of $H_i\left(\mathbb{T}^d\right)$ and the action of the symmetry group of $\mathbb{T}_N^d$ on the homology. $H_i\left(\mathbb{T}^d\right)$ is isomorphic to the group of degree i homogeneous polynomials $x_1, \ldots, x_d$, where each x_j corresponds to the image of one of the S^1 factors into the product $\mathbb{T}^d = S^1 \times \ldots \times S^1$ under the induced map on homology $H_1(S^1) \to H_1(\mathbb{T}^d)$. (This follows from the Künneth formula for homology; see, for example, Section 3b of [12].) The standard basis elements for $H_i(\mathbb{T}^d)$ are the degree i monomials in $x_1, \ldots, x_d$, denoted $\alpha_1, \ldots, \alpha_D$.

The symmetry group G of $\mathbb{T}_N^d$ is generated by permutations of the S^1 factors ("rotations" which act on $H_i(\mathbb{T}^d)$ by permuting $x_1, \ldots, x_d$), reflections which reverse the orientation of one of the S^1 factors (and send x_j to $-x_j$ for some j), and translations (which act trivially on the homology). G acts transitively on the plaquettes of $\mathbb{T}_N^d$. (The rotations and reflections generate the hyperoctahedral group B_d, the group of symmetries of the d-dimensional cube.)

3. Surjectivity

The goal of this section is to show that if $p > p_c$ then $\mathbb{P}_p(S) \to 1$ as $N \to \infty$, where

$$p_c = p_c(i, d) = \inf\left\{p : \liminf_{N \to \infty} \mathbb{P}_p(A) > 0\right\} .$$

First, we will prove that $\mathbb{P}_p(S) \geq b_0 \mathbb{P}_p(A)^{b_1}$ for some $b_0, b_1 > 0$ that do not depend on N. The argument is somewhat involved, but is inspired by the observation that if $\sigma \neq 0 \in H_i(\mathbb{T}^d)$ and G is the symmetry group of $\mathbb{T}^d$, then $G\sigma$ spans $H_i(\mathbb{T}^d)$ (that is, $H_i(\mathbb{T}^d)$ is an irreducible representation of G). If we could bound the probability that $\sigma \in \operatorname{im} \phi_*$ away from zero for some particular σ, the proof would be straightforward.

As a preliminary step, we will show the corresponding result for the event that $\alpha_k \in \operatorname{im} \phi_*$ for some k, where $\alpha_1, \ldots, \alpha_D$ are the standard basis elements of $H_i(\mathbb{T}^d)$. For $\sigma \in H_i(\mathbb{T}^d)$ let $|\sigma|$ be the number of non-zero terms appearing when σ is written as a linear combination of the α_k's, let C_m be the event that there exists a $\sigma \in H_i(\mathbb{T}^d)$ with $|\sigma| = m$ and let $M = \min\{|\sigma|, \sigma \neq 0 \in \operatorname{im} \phi_*\}$. Using the action of reflections on the $H_i(\mathbb{T}^d)$, we will show that we can bound $\mathbb{P}_p(M \leq m-1)$ below in terms of $\mathbb{P}_p(C_m)$ and consequently in terms of $\mathbb{P}_p(M \leq m)$. Recursively, we obtain a lower bound for $\mathbb{P}_p(M = 1)$ involving $\mathbb{P}_p(M \leq D) = \mathbb{P}_p(A)$.

Lemma 9. *There exist constants $c_0 = c_0(D) > 0$ and $c_1 = c_1(D) > 0$ that do not depend on N so that*

$$\mathbb{P}_p(M = 1) \geq c_0 \mathbb{P}_p(A)^{c_1} .$$

Proof. We will begin by finding a lower bound for $\mathbb{P}_p(M \leq m-1)$ in terms of $\mathbb{P}_p(C_m)$. For $J = \{j_1, \ldots, j_m\} \subset [D]$ let B_J be the event that

$$\sum_{k=1}^{m} a_k \alpha_{j_k} \in \operatorname{im} \phi_*$$

for some $a_1, \ldots, a_m \in \mathbb{R} \setminus \{0\}$. C_m is the union of the events B_J as J ranges over all subsets of $[D]$ of size m and there are $\binom{D}{m}$ such subsets so there exists a J so that

$$\mathbb{P}_p(B_J) \geq \binom{D}{m}^{-1} \mathbb{P}_p(C_m) \,. \tag{2}$$

For convenience, renumber the monomials so that $J = \{1, \ldots, m\}$. Let B_J^δ be defined as B_J with the additional requirement that $\delta a_1 a_2 > 0$, for $\delta \in \{-1, 1\}$. α_1 and α_2 are distinct monomials so there exists a variable x_r appearing in α_2 but not α_1. If ρ is the reflection of $\mathbb{T}^d$ sending x_r to $-x_r$ then $B_J^{-1} = \rho B_J^1$ and $B_J = B_J^1 \cup B_J^{-1}$. It follows that

$$\mathbb{P}_p\left(B_J^1\right) = \mathbb{P}_p\left(B_J^{-1}\right) \geq \frac{1}{2}\mathbb{P}_p(B_J) \,. \tag{3}$$

If both B_J^1 and B_J^{-1} occur then there exist non-zero $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ with $a_1 a_2 > 0$ and $b_1 b_2 < 0$ so that $\sigma_1 := \sum_{k=1}^m a_k \alpha_k$ and $\sigma_2 := \sum_{k=1}^m b_k \alpha_k$ are in the image of ϕ_*. Then $\sigma_1 - \frac{a_2}{b_2}\sigma_2$ is also in the image of ϕ_*, it is non-zero because a_1 and $\frac{a_2}{b_2} b_1$ have opposite signs, and it is the sum of at most $m-1$ basis elements of $H_1\left(\mathbb{T}^d\right)$. Therefore, $B_J^1 \cap B_J^{-1} \subseteq C_m \cap \{M \leq m-1\}$ and

$$\begin{aligned}
\mathbb{P}_p\left(C_m \cap \{M \leq m-1\}\right) \geq\ & \mathbb{P}_p\left(B_J^1 \cap B_J^{-1}\right) & \\
\geq\ & \mathbb{P}_p\left(B_J^1\right)\mathbb{P}_p\left(B_J^{-1}\right) & \text{by Harris's Inequality} \\
\geq\ & \frac{1}{4}\mathbb{P}_p(B_J)^2 & \text{by Eqn. 3} \\
\geq\ & \left(2\binom{D}{m}\right)^{-2} \mathbb{P}_p(C_m)^2 & \text{by Eqn. 2} \\
:=\ & \hat{c}(m)\,\mathbb{P}_p(C_m)^2 \,. &
\end{aligned}$$

To convert this into an inequality involving $\mathbb{P}_p(M \leq m-1)$ and $\mathbb{P}_p(M \leq m)$, note that $\hat{c}(m) \leq \frac{1}{4}$ so if $0 < \alpha, \beta < 1$ then

$$\hat{c}(m)(\alpha+\beta)^2 < \hat{c}(m)\,\alpha^2 + \beta \,. \tag{4}$$

We have that

$$\begin{aligned}\mathbb{P}_p\left(M \leq m-1\right) = &\ \mathbb{P}_p\left(C_m \cap \{M \leq m-1\}\right) + \mathbb{P}_p\left(C_m^c \cap \{M \leq m-1\}\right) \\ \geq &\ \hat{c}\left(m\right)\mathbb{P}_p\left(C_m\right)^2 + \mathbb{P}_p\left(C_m^c \cap \{M \leq m-1\}\right) \\ \geq &\ \hat{c}\left(m\right)\left[\mathbb{P}_p\left(C_m\right) + \mathbb{P}_p\left(C_m^c \cap \{M \leq m-1\}\right)\right]^2 \qquad \text{by Eqn. 4} \\ = &\ \hat{c}\left(m\right)\mathbb{P}_p\left(M \leq m\right)^2 .\end{aligned}$$

Applying the previous computation $D-1$ times to $A = \{M \leq D\}$ yields

$$\begin{aligned}\mathbb{P}_p\left(M = 1\right) \geq &\ \left[\prod_{m=1}^{D-1} \hat{c}\left(m\right)^{2^{m-2}}\right]\mathbb{P}_p\left(A\right)^{2^{D-1}} \\ = &\ \left[\prod_{m=1}^{D-1}\left(2\binom{D}{m}\right)^{-2^{m-1}}\right]\mathbb{P}_p\left(A\right)^{2^{D-1}} \\ := &\ c_0\mathbb{P}_p\left(A\right)^{c_1} .\end{aligned}$$

□

Next, we use the action of rotations on $H_i\left(\mathbb{T}^d\right)$ to convert the bound in the previous lemma into a bound on the probability that ϕ_* is surjective.

Lemma 10. *There exist constants* $b_0 = b_0\left(D\right) > 0$ *and* $b_1 = b_1\left(D\right) > 0$ *that do not depend on* N *so that*

$$\mathbb{P}_p\left(S\right) \geq b_0\mathbb{P}_p\left(A\right)^{b_1} .$$

Proof. Let $E_k = \{\alpha_k \in \operatorname{im}\phi_*\}$ for $1 \leq k \leq D$. By the transitivity of the symmetry action of $\mathbb{T}^d$ on the α_k's, we have that $\mathbb{P}_p\left(E_j\right) = \mathbb{P}_p\left(E_k\right)$ for all k and j. As $\{M = 1\} = \cup_{k=1}^{D} E_k$, it follows from Lemma 9 that

$$\mathbb{P}_p\left(E_k\right) \geq \frac{1}{D}\mathbb{P}_p\left(M = 1\right) \geq \frac{1}{D}c_0\mathbb{P}_p\left(A\right)^{c_1}$$

for each k. Also, $S = \cap_k E_k$ because $\alpha_1, \ldots, \alpha_D$ span $H_i\left(\mathbb{T}^d\right)$. The events E_k are increasing so Harris's Inequality yields that

$$\begin{aligned}\mathbb{P}_p\left(S\right) = &\ \mathbb{P}_p\left(\cap_k E_k\right) \\ \geq &\ \prod_{k=1}^{D}\frac{1}{D}c_0\mathbb{P}_p\left(A\right)^{c_1} \\ = &\ D^{-D}c_0^D\mathbb{P}_p\left(A\right)^{Dc_1} \\ := &\ b_0\mathbb{P}_p\left(A\right)^{b_1} .\end{aligned}$$

□

We combine the previous lemma with Theorem 5 to show that the probability of S goes to 1 if $p > p_c(i,d)$.

Proposition 11. *If $p > p_c(i,d)$ then*

$$\mathbb{P}_p(S) \to 1$$

as $N \to \infty$.

Proof. Let $p > q > p_c(i,d)$, let G be the symmetry group of $\mathbb{T}^d$, and let X be the set of i-plaquettes of $\mathbb{T}^d_N$. By the definition of $p_c(i,d)$, there exists an $r > 0$ so that $\mathbb{P}_q(A) > r > 0$ for all sufficiently large N. As such, Lemma 10 implies that $\mathbb{P}_q(S) > b_0 r^{b_1} > 0$ for all sufficiently large N. Choose an ϵ between 0 and $b_0 r^{b_1}$. S is increasing and invariant under the action of G and G acts transitively on X so the hypotheses of Theorem 5 are met. Re-arranging Equation 1 gives that $\mu_p(S) > 1-\epsilon$ when

$$\log(|X|) > \frac{\rho \log(1/(2\epsilon))}{p-q}.$$

To compute $|X|$, note that each cube of $\mathbb{T}^d_N$ has $2^{d-i}D$ i-faces and each i-face is adjacent to 2^{d-i} cubes. There are N^d cubes so

$$|X| = N^d \binom{d}{i}.$$

Combining the two previous equations yields that $\mu_p(S) > 1-\epsilon$ whenever

$$\log(N) \geq \frac{1}{d}\left(\frac{\rho \log(1/(2\epsilon))}{p-q} - \log(D)\right),$$

or equivalently

$$N \geq e^{-\frac{D}{d}} 2^{-\frac{\rho}{d(p-q)}} \epsilon^{-\frac{\rho}{d(p-q)}}.$$

□

4. The Case $d = 2i$

We now prove Theorem 1, that $p_c(i,2i) = 1/2$ is a sharp threshold for A when $d = 2i$.

Proof of Theorem 1. Half-dimensional plaquette percolation is self-dual so

$$\mathbb{P}_{1/2}(A) = \mathbb{P}_{1/2}(A^\bullet).$$

By Lemma 8 at least one of the events A and $A^{\bullet}$ must occur. Therefore,

$$2\mathbb{P}_{1/2}\left(A\right)=\mathbb{P}_{1/2}\left(A\right)+\mathbb{P}_{1/2}\left(A^{\bullet}\right)\geq 1$$

and

$$\mathbb{P}_{1/2}\left(A\right)\geq 1/2$$

for all N. It follows that $p_c \leq 1/2$. Thus, if $p > 1/2$ then

$$\mathbb{P}_p\left(S\right)\to 1$$

as $N \to \infty$ by Proposition 11. On the other hand, if $p < 1/2$ then $1-p > 1/2$ so

$$\mathbb{P}_p\left(S^{\bullet}\right)=\mathbb{P}_{1-p}\left(S\right)\to 1$$

by self-duality and

$$\mathbb{P}_p\left(A\right)=1-\mathbb{P}_p\left(S^{\bullet}\right)\to 0$$

by Lemma 8, as $N \to \infty$.

□

5. Sharpness and Duality

In this section, we combine the Duality Lemma (Lemma 8) with Proposition 11 to examine the behavior of $\mathbb{P}_p\left(A\right)$ below $q_c\left(i,d\right)$ and above $p_c\left(i,d\right)$. We also relate these thresholds to $p_c\left(d-i,d\right)$ and $q_c\left(d-i,d\right)$. We remind the reader that

$$q_c\left(i,d\right)=\sup\left\{p:\limsup_{N\to\infty}\mathbb{P}_p\left(S\right)<1\right\}.$$

First, Proposition 11 from the previous section has the following corollary.

Corollary 12. $q_c\left(i,d\right)\leq p_c\left(i,d\right)$.

Next, we use the same proposition with duality to understand the behavior of $\mathbb{P}_p\left(A\right)$ below $q_c\left(i,d\right)$.

Proposition 13. *If* $p < q_c\left(i,d\right)$ *then*

$$\mathbb{P}_p\left(A\right)\to 0$$

as $N \to \infty$.

Proof. Let $p < q < q_c\left(i,d\right)$. By the definition of $q_c\left(i,d\right)$ there exists an $s > 0$ so that $\mathbb{P}_q\left(S\right) < s < 1$ for all sufficiently large N. Then, Lemma 8 implies that $\mathbb{P}_q\left(A^{\bullet}\right) > 1-s > 0$ for all sufficiently large N. It follows that

$$1-p > 1-q \geq p_c\left(d-i,d\right).$$

$P^{\bullet} = P^{\bullet}(i, d, N, p)$ has the same distribution as $P(d-i, d, N, 1-p)$ so, by Proposition 11,

$$\mathbb{P}_p(S^{\bullet}) \to 1$$

as $N \to \infty$. A final application of Lemma 8 yields

$$\mathbb{P}_p(A) = 1 - \mathbb{P}_p(Z) = 1 - \mathbb{P}_p(S^{\bullet}) \to 0$$

as $N \to \infty$. □

A similar argument shows a partial duality result for any i and d.

Proposition 14.

$$p_c(i, d) + q_c(d-i, d) = 1\,.$$

Proof. Let $p > p_c(i, d)$. Then

$$\begin{aligned} \mathbb{P}_p(A^{\bullet}) &= 1 - \mathbb{P}_p(Z^{\bullet}) && \text{by definition} \\ &= 1 - \mathbb{P}_p(S) && \text{by Lemma 8} \\ &\to 0 && \text{by Proposition 11} \end{aligned}$$

as $N \to \infty$. Therefore, $1 - p \leq q_c(d-i, d)$ for all $p > p_c(i, d)$ and

$$p_c(i, d) + q_c(d-i, d) \geq 1\,. \tag{5}$$

Until now, we have suppressed the dependence of probabilities of events on N. To work with subsequences in this argument, denote the probability of an event B for $P(i, d, N, p)$ by $\mathbb{P}_{p,N}(B)$.

Let $p < p_c(i, d)$. Then there is a subsequence $\{n_1, n_2, \ldots\}$ of $\mathbb{N}$ for which

$$\mathbb{P}_{p,n_k}(A) \to 0\ .$$

By Lemma 8,

$$\mathbb{P}_{p,n_k}(S^{\bullet}) \to 1$$

so

$$\limsup_{N \to \infty} \mathbb{P}_p(S^{\bullet}) = 1$$

and $1 - p \geq q_c(i, d)$ for all $p < p_c(i, d)$. Therefore,

$$p_c(i, d) + q_c(d-i, d) \leq 1$$

which holds with equality by Equation 5. □

Propositions 11, 13, and 14 show that duality between $p_c(i,d)$ and $p_c(d-i,d)$ is equivalent to the existence of a sharp threshold for A. We say that r_c is a **sharp threshold** for an event B if

$$\begin{cases} \mathbb{P}_p(B) \to 0 & p < r_c \\ \mathbb{P}_p(B) \to 1 & p > r_c \end{cases}.$$

Corollary 15. *The following are equivalent.*

(a) $p_c(i,d)$ *is a sharp threshold for* A.

(b) $p_c(i,d)$ *is a sharp threshold for* S.

(c) $p_c(i,d) = q_c(i,d)$.

(d) $p_c(i,d) + p_c(d-i,d) = 1$.

In the next section, we demonstrate the existence of a sharp threshold for the cases $i = 1$ and $i = d-1$.

6. The Cases $i = 1$ and $i = d - 1$

We show that $p_c(1,d)$ and $q_c(1,d)$ coincide and equal the critical threshold for bond percolation on $\mathbb{Z}^d$, denoted here by $\hat{p}_c = \hat{p}_c(d)$. To do so, we rely on two results from the classical theory of this system in the subcritical and supercritical regimes. In the former regime, we use Menshikov's Theorem [14] showing an exponential decay in the radius of the cluster at the origin.

For a vertex x and a subset S of $\mathbb{T}_N^d$, denote the event that x is connected to a vertex in S by a path of edges in P by $x \leftrightarrow S$.

Theorem 16 (Menshikov's Theorem)**.** *Consider bond percolation on* $\mathbb{Z}^d$. *If* $p < \hat{p}_c$ *then there exists a* $\kappa(p) > 0$ *so that*

$$\mathbb{P}_p\left(0 \leftrightarrow \partial[0,M]^d\right) \le e^{-\kappa(p)M}$$

for all $M > 0$.

We apply Menshikov's Theorem to show that the probability of a giant one-cycle limits to zero as $N \to \infty$ when $p < \hat{p}_c$.

Proposition 17. $q_c(1,d) \ge \hat{p}_c$

Proof. Let $p < \hat{p}_c$ and consider the the cube $C = [\lfloor N/3 \rfloor, \lceil 2N/3 \rceil]^d \subset \mathbb{T}_N^d$. Let B_0 be the event that a giant 1-cycle of P passes through C. That is, there is a connected 1-cycle σ of P that contains an edge of C and satisfies $\phi_*([\sigma]) \neq 0$

where $[\sigma]$ denotes the homology class of σ. A cycle must contain at least one edge and the proportion of edges of $\mathbb{T}_N^d$ that intersect C is at least $(1/3)^d$ so, by translation invariance,

$$\mathbb{P}_p\left(B_0\right) \geq \frac{1}{3^d}\mathbb{P}_p\left(A\right) .$$

Define B_1 to be the event that there is an vertex in $x \in \partial C$ so that $x \leftrightarrow \partial\left[0, N-1\right]^d$. A 1-cycle contained in the interior of $\left[0, N-1\right]^d$ is null–homologous in $\mathbb{T}^d$ so $B_0 \implies B_1$. Combining this with the previous equation yields

$$\mathbb{P}_p\left(A\right) \leq 3^d\mathbb{P}_p\left(B_0\right) \leq 3^d\mathbb{P}_p\left(B_1\right) . \tag{6}$$

Let F be the collection of vertices of $\mathbb{T}_d^N$ in ∂C, and let X be the number of vertices in F that are connected to $\partial\left[0, N-1\right]^d$ by a path of bonds in P. Our strategy is to bound the probability of B_1 by way of a bound on X. First, if $x \in F$ then Theorem 16 implies that

$$\begin{aligned} \mathbb{P}_p\left(x \leftrightarrow \partial\left[0, N-1\right]^d\right) \leq\ & \mathbb{P}_p\left(0 \leftrightarrow \partial\left[0, N-1-\lceil\frac{2N}{3}\rceil\right]^d\right) \\ \leq\ & e^{-(N/3-2)\kappa(p)} . \end{aligned} \tag{7}$$

Next, we bound the number of vertices in F. The number of vertices in the boundary of a d-dimensional cube of width M is bounded above by the $2d$ times the number of vertices in each of its $d-1$-faces (where we have double-counted vertices in the $d-2$-faces), which in turn equals $2dM^d$. It follows that

$$\begin{aligned} \left|F\right| \leq\ & 2d\left(\lceil\frac{2N}{3}\rceil - \lfloor\frac{N}{3}\rfloor\right)^d \\ \leq\ & 2d\left(\frac{N}{3}-2\right)^d . \end{aligned} \tag{8}$$

By our previous computations,

$$\begin{aligned} \mathbb{E}\left(X\right) =\ & \sum_{x \in \partial C}\mathbb{P}\left(x \leftrightarrow \partial\left[0, N-1\right]^d\right) \\ \leq\ & \left|F\right| e^{-(N/3-2)\kappa(p)} && \text{by Eqn. 7} \\ \leq\ & 2d\left(\frac{N}{3}-2\right)^d e^{-(N/3-2)\kappa(p)} && \text{by Eqn. 8} . \end{aligned} \tag{9}$$

Finally,

$$\begin{aligned}\mathbb{P}_p(A) &\leq 3^d\mathbb{P}_p(B_1) && \text{by Eqn. 6}\\ &= 3^d\mathbb{P}_p(X\geq 1)\\ &\leq 3^d\mathbb{E}_p(X) && \text{by Markov's Inequality}\\ &\leq 3^d 2d\left(\frac{N}{3}-1\right)^d e^{-(N/3-2)\kappa(p)} && \text{by Eqn. 9}\end{aligned}$$

which goes 0 as $N\to\infty$. □

In the supercritical regime, we use the following lemma on crossing probabilities inside a rectangle (which is Lemma 7.78 in [9]).

Lemma 18. *Let $p>\hat{p}_c$. Then there is an $L>0$ and a $\delta>0$ so that if $N>0$ and $x\in[0,N-1]^{d-1}\times[0,L]$, then probability that 0 is connected to x inside $P\cap\left([0,N-1]^{d-1}\times[0,L]\right)$ is at least δ.*

Proposition 19. $p_c(1,d)\leq\hat{p}_c$

Proof. Let $p>\hat{p}_c$, and let B be the event that there is a path of edges of P connecting 0 to $(N-1)\boldsymbol{e}_1=(N-1,0,\ldots,0)$ inside of $[0,N-1]^d$. By the previous lemma, there is a $\delta>0$ so that $\mathbb{P}_p(B)\geq\delta$.

If B occurs, then path obtained by adding the edge between $(N-1)\boldsymbol{e}_1$ and $N\boldsymbol{e}_1=0$ is a giant 1-cycle. It follows that

$$\mathbb{P}_p(A)\geq p\mathbb{P}_p(B)\geq p\delta$$

for any choice of N. Therefore, $p_c(1,d)\leq p$ for all $p\geq\hat{p}_c$ and $p_c(1,d)\leq\hat{p}_c$. □

The proof of Theorem 2 is completed by combining Propositions 17 and 19 with Corollary 15.

Proof of Theorem 2. Propositions 17 and 19 show that

$$p_c(1,d)\leq\hat{p}_c\leq q_c(1,d)\ .$$

However, $q_c(1,d)\leq p_c(1,d)$ by Corollary 12 so

$$p_c(1,d)=q_c(1,d)=\hat{p}_c\,.$$

Therefore, it follows from Corollary 15 that $\hat{p}_c$ and $1-\hat{p}_c$ are sharp thresholds for 1-dimensional and $(d-1)$-dimensional percolation on the $\mathbb{T}^d$, respectively. □

7. The General Case

Next, we prove that the critical probabilities $p_c\left(i,d\right)$ are strictly increasing in i and strictly decreasing in d. We do so comparing percolation on $\mathbb{T}_N^d$ with the thickened $d-1$-dimensional slice $\mathbb{T}_N^d \cap \{0 \leq x_1 \leq 1\}$. This will complete the proof of Theorem 3. Compare the first part of the proof to that of Lemma 4.9 of [3].

Proposition 20. *For* $0 < i < d-1$,

$$p_c\left(i,d\right) < p_c\left(i,d-1\right) < p_c\left(i+1,d\right) \,.$$

Proof. First, we will show that

$$p_c\left(i,d\right) \leq p_c\left(i,d-1\right) \leq p_c\left(i+1,d\right) \,. \tag{10}$$

Let $T = \mathbb{T}_N^d \cap \{x_1 = 0\}$. T is a torus of dimension $d-1$ and, by a standard argument, the map on homology $\alpha_* : H_j\left(T\right) \to H_j\left(\mathbb{T}^d\right)$ induced by the inclusion $T \hookrightarrow \mathbb{T}^d$ is injective for all j. $P \cap T$ is distributed as $P\left(i,d-1,N,p\right)$.

Define A_{d-1} to be the event that $\gamma_* : H_i\left(P\cap T\right) \to H_i\left(T\right)$ is non-zero, where γ_* is induced by the inclusion $P\cap T \hookrightarrow T$. If A_{d-1} holds then $\alpha_* \circ \phi_*$ is also non-zero, as α_* is injective. But $\alpha_* \circ \gamma_* = \phi_* \circ \beta_*$, where β_* is the map on homology $\beta_* : H_i\left(P\cap T\right) \to H_i\left(P\right)$ induced by the inclusion $P\cap T \hookrightarrow P$, so ϕ_* is also non-zero. It follows that $A_{d-1} \implies A$. Therefore $p_c\left(i,d-1\right) \geq p_c\left(i,d\right)$ by the definition of that threshold.

Observe that $H_i\left(\mathbb{T}^d\right)$ is generated by the images of the maps on homology $H_i\left(\mathbb{T}^d \cap \{x_j = 0\}\right) \to H_i\left(\mathbb{T}^d\right)$ induced by the inclusions $\mathbb{T}^d \cap \{x_j=0\} \hookrightarrow \mathbb{T}^d$ as j ranges from 1 to d. (The image of the j-th map contains all monomials that do not include x_j as a factor; see the discussion at the end of Section 2) Denote by S_j the event that the map $H_i\left(P \cap \{x_j=0\}\right) \to H_i\left(\mathbb{T}^d \cap \{x_j = 0\}\right)$ induced by inclusion is surjective and let $q > q_c(i,d-1)$. Then there is a subsequence $(n_1,n_2,\ldots)$ of $\mathbb{N}$ so that

$$\mathbb{P}_{p,n_k}\left(S_j\right) \to 1$$

as $k \to \infty$ for $j = 1,\ldots,d$. As $S \subset \cap_j S_j$, Harris's Inequality implies that $\mathbb{P}_{p,n_k}\left(S\right) \to 1$ also. Therefore, $p > q_c\left(i,d\right)$ and $q_c\left(i,d-1\right) \geq q_c\left(i,d\right)$. Then, from Proposition 14 we obtain

$$p_c(i,d-1) = 1 - q_c(d-i-1,d-1) \leq 1 - q_c(d-i-1,d) = p_c(i+1,d) \tag{11}$$

which shows Equation 10.

It will be useful later in the argument to observe that these inequalities, together with Theorem 2 and known lower bounds on $\hat{p}_c$ (see [4], for example),

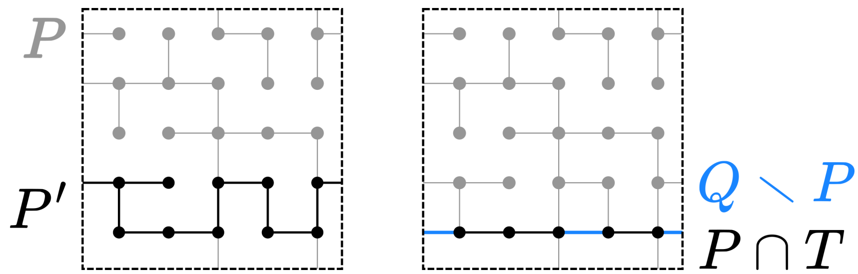

FIGURE 5. The setup in the proof of Proposition 20 for the case $d = 2, i = 1$. On the left, P' is shown in black and the remaining faces of P are depicted in gray. On the right, $P \cap T$ is in black and the additional faces of Q are shown in blue. Note that giant cycles exist in P, P', and Q, but not in $P \cap T$.

imply that

$$0 < p_c(1, d) \leq p_c(i, d) \leq p_c(i, i+1) < 1 \,. \tag{12}$$

Furthermore, we can show $p_c(i, d) < p_c(i, d-1)$ using the thicker cross-section $T' = \mathbb{T}_N^d \cap \{0 \leq x_1 \leq 1\}$. Note that an i-face v of T is in the boundary of a unique $i+1$-face $w(v)$ of T' that is not contained in T (for example, if $v = \{0\} \times [0,1]^i \times \{0\}^{d-i-1}$, then $w = [0,1]^{i+1} \times \{0\}^{d-i-1}$). The idea is to sometimes add v to T when the other i-faces of w are present, effectively increasing the percolation probability in T by a small amount. However, we must be careful to do so in a way so that the i-faces remain independent.

The i-faces of T' are divided into three subsets: those included in T, those which are perpendicular to T (that is, i-faces not included in T which intersect T in their boundary), and those parallel to T (that is, i-faces of the form $v + \boldsymbol{e}_1$, where v is an i-face of T and $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$). For an i-face v of T, let $J(v)$ be the set of all perpendicular i-faces that meet v at an $i-1$ face. v, $v + \boldsymbol{e}_1$, and $J(v)$ are the i-faces of the $i+1$-face $w(v)$. Also, for a perpendicular i-face u of T', let $K(u) = \{v : u \in J(v)\}$. Note that for any u and v

$$|J(v)| = 2i \qquad \text{and} \qquad |K(u)| = 2(d-i) \,. \tag{13}$$

We define a coupling between i-dimensional plaquette percolation P' on T' with probability p and i-dimensional percolation Q with probability $p + p(1-p)q^{2i}$ on T, where $q = q(p)$ is chosen to satisfy $p = 1 - (1-q)^{2(d-i)}$). For all pairs (v, u) where v is an i-face of T and $u \in J(v)$, define independent Bernoulli random variables $\kappa(u, v)$ to be 1 with probability q and 0 with probability $1-q$. Let $P' \subset T'$ be the subcomplex containing the $i-1$-skeleton of T' where each i-face in T or parallel to T is included independently with probability

p, and the other i-faces u of T' are included if $\kappa(u,v)=1$ for at least one $v\in K(u)$. Observe that

$$\mathbb{P}(u\in P')=1-\mathbb{P}\left(\cap_{v\in K(u)}\{\kappa(u,v)=0\}\right)=1-(1-q)^{2(d-i)}=p$$

(using Equation 13), and that the faces u are included independently. That is, P' is percolation with probability p on T'. On the other hand, define $Q\subset T$ by starting with all faces of $P'\cap T$ and adding an i-face $v\notin P'$ if $v+\boldsymbol{e}_1\in P'$ and $\kappa(v,u)=1$ for all $u\in J(v)$. Then Q is percolation on T with probability $p+p(1-p)q^{2i}>p$. See Figure 5.

As $p+p(1-p)q^{2i}$ is a continuous function of p and $0<p_c(i,d-1)<1$ (Equation 12), we can choose p to satisfy

$$0<p<p_c(i,d-1)<p+p(1-p)q^{2i}<1\,.$$

Then

$$\mathbb{P}_{p+p(1-p)q^{2i}}\left(\xi_*\text{ is non-trivial}\right)\to 1 \tag{14}$$

as $N\to\infty$ by the definition of $p_c(i,d-1)$, where $\xi_*:H_i(Q)\to H_i(T)$ is the map on homology induced by the inclusion $Q\hookrightarrow T$.

Extend P' to plaquette percolation P on all of $\mathbb{T}_N^d$ by including the i-faces in $\mathbb{T}_N^d\setminus T'$ independently with probability p. If σ is an i-cycle of Q we can write

$$\sigma=\sum_j a_ju_j+\sum_k b_kv_k$$

where $u_j\notin P$ and $v_k\in P$ for all j and k. Then, by construction, we can form a corresponding i-cycle σ' of P by setting

$$\sigma'=\sigma+\sum_j a_j\partial w\left(u_j\right)\,.$$

σ and σ' are homologous in $\mathbb{T}^d$, so $\alpha_*\circ\xi_*([\sigma])=\phi_*([\sigma'])$ In particular, if ξ_* is non-trivial then ϕ_* is non-trivial as well. Using Equation 14, it follows that

$$\mathbb{P}_p(A)\geq\mathbb{P}_{p+p(1-p)q^{2i}}\left(\xi_*\text{ is non-trivial}\right)\to 1\,,$$

as $N\to\infty$. Therefore,

$$p_c(i,d)\leq p<p_c(i,d-1)\,.$$

Applying reasoning analogous to that in the paragraph preceding Equation 11 to the cross-sections $\mathbb{T}_N^d\cap\{0\leq x_j\leq 1\}$, we can also show $q_c(i,d)<q_c(i,d-1)$. Then from Proposition 14 we obtain

$$p_c(i,d-1)=1-q_c(d-i-1,d-1)<1-q_c(d-i-1,d)=p_c(i+1,d)\,.$$

□

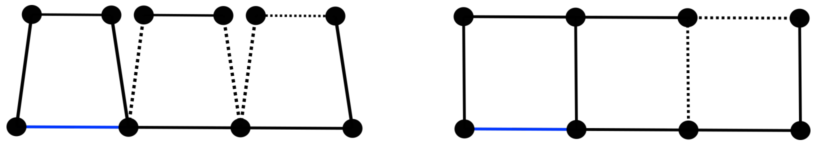

FIGURE 6. Percolation on T'' (left) mapping to percolation on T' (right) for $i = 1, d = 2$. The blue edges are in $Q \setminus P$.

An alternative approach to the proof of Proposition 20 is to construct a third space T'' by attaching a new $i+1$-cube to each i-face of T along one of the cube's i-faces. We can define inhomogeneous percolation P'' on T'' by starting with the $i-1$-skeleton of T'', adding each i-face of T and and each i-face parallel to T independently with probability p, and adding the perpendicular i-faces independently with probability q (these faces play the same role as the random variables $\kappa(u,v)$ above). Giant cycles in P'' are ones that are mapped non-trivially to $H_i(T'')$ by the map on homology induced by the inclusion $P'' \hookrightarrow T''$, and they appear at a lower value of p than $p_c(i, d-1)$ (precisely when they appear in Q as defined above). The proof is finished by observing that the quotient map $\pi : T'' \to T'$ identifying the corresponding perpendicular faces of neighboring cubes induces an injective map on homology, and therefore the existence of giant cycles in P'' implies the existence of giant cycles in P. This idea is illustrated in Figure 6. Note that our definition of giant cycles in T'' can be adapted to give a more general notion of homological percolation in the i-skeleton of a cubical or simplicial complex whose i-dimensional homology is nontrivial.

Proof of Theorem 3. By Equation 12, $q_c(i,d), p_c(i,d) \in (0,1)$. The remaining statements follow from Propositions 11, 13, 14, and 20. □

Note that we could alternatively show that $p_c(i,d), q_c(i,d) \in (0,1)$ by modifying the proof of Proposition 17 to work for the lattice of i-plaquettes in $\mathbb{Z}^d$ and using a Peierls-type argument to obtain the bound

$$\frac{1}{2d-i+1} \leq q_c(i,d) \leq p_c(i,d) \leq 1 - \frac{1}{d+i+1}.$$

8. FUTURE DIRECTIONS

It seems that not much is known about percolation with higher-dimensional cells or homological analogues of bond or site percolation.

- Our arguments are for homology with $\mathbb{Q}$ coefficients, and can easily be extended to fields of characteristic zero or finite fields of odd characteristic. However, we do not know yet whether it is possible for the thresholds p_c to depend on choice of coefficient ring more generally.
- Are there scaling limits for plaquette percolation? For bond percolation in the plane at criticality, conjecturally we get SLE. This could be a reasonable question to approach experimentally.
- Is there a limiting distribution for $\operatorname{rank}\phi_*$, as $N \to \infty$? When $d = 2i$, our results imply that the distribution is symmetric and the expectation satisfies $\mathbb{E}[\operatorname{rank}\phi_*] = \binom{d}{d/2}/2$, but at the moment we do not know anything else.
- One of the most interesting possibilities we can imagine would be a generalization of the Harris–Kesten theorem when $d = 2i$, on the whole lattice $\mathbb{Z}^d$ rather than on the torus $\mathbb{T}_N^d$. One possibility might be to compactify $\mathbb{R}^d$ to a torus T^d. In various proofs of the Harris–Kesten theorem, a key step is to go from crossing squares to crossing long, skinny rectangles—see, for example, Chapter 3 of [4]. One difficulty is that we do not currently have a high-dimensional version of the Russo–Seymour–Welsh method, passing from homological "crossings" of high-dimensional cubes to long, skinny boxes.

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