Small Open Economy RBC Model

Uribe, Chapter 4

1 Basic Model

1.1 Uzawa Utility

$$E_0 \sum_{t=0}^{\infty} \theta_t U\left(c_t, h_t\right)$$

$$\theta_0 = 1$$

$$\theta_{t+1} = \beta\left(c_t, h_t\right)\theta_t; \quad \beta_c < 0; \quad \beta_h > 0.$$

- Time-varying discount factor

- With a constant discount factor, consumption and net foreign assets are non-stationary
  * Rise permanently with an increase in productivity
  * Unit roots imply no steady state
  * Cannot claim that the linearized model, which converges to a steady state, behaves as the original model
  * Cannot compute unconditional moments

– Discount factor is decreasing in wealth

  * As wealth increases, c increases and h decreases reducing θ_{t+1} below θ_t

  * Equivalently, agents become less patient as wealth increases

  * Positive productivity shock raises wealth and consumption and reduces hours

  * Therefore, θ falls, increasing consumption, sending wealth back down

  * Get a steady state value of wealth with this discount factor and therefore a steady state

1.2 Household Budget Constraint

- Write in terms of household debt (d) instead of assets

$$d_t = (1 + r_{t-1})\, d_{t-1} - y_t + c_t + i_t + \Phi\left(k_{t+1} - k_t\right)$$

- Cost of Investment includes capital adjustment costs $\Phi\left(k_{t+1} - k_t\right)$
  - Adjustment cost reduce the volatility of investment
  - Adjustment costs is increasing in the size of the change in capital, reducing the optimal size of investment
  - Restrictions assure that when capital is fixed in the steady state, there are no investment costs

$$\Phi(0) = \Phi'(0) = 0$$

- Output

$$y_t = A_t F(k_t, h_t)$$

- Evolution of capital

$$k_{t+1} = i_t + (1 - \delta) k_t$$

1.3 Optimization

- Households choose $\{c_t, h_t, k_{t+1}, d_t, \theta_{t+1}\}$
  - Subject to above constraints
  - and NPG condition requiring the present value of debt in the limit to be non-positive

$$\lim_{j\to\infty} E_t \frac{d_{t+j}}{\prod_{s=1}^{j}(1+r_s)} \leq 0$$

  - Substitute for investment in household budget constraint

$$d_t = (1 + r_{t-1})\, d_{t-1} - y_t + c_t + k_{t+1} - (1-\delta)\, k_t + \Phi\,(k_{t+1} - k_t)$$

- Lagrange multipliers
  - $\theta_t \lambda_t$ on household budget constraint
  - η_t on evolution of discount factor

$$\theta_{t+1} = \beta\left(c_t, h_t\right)\theta_t$$

1.4 First Order Conditions

- d_t

$$\lambda_t = \beta(c_t, h_t) E_t \lambda_{t+1} (1 + r)$$

  - $\beta(c_t, h_t)$ is the ratio of two discount factors

$$1 = \frac{\beta(c_t, h_t) E_t \lambda_{t+1} (1 + r)}{\lambda_t}$$

  - if $\beta(1 + r)$ is fixed at unity a wealth-reducing shock which raises λ_t raises $E_t \lambda_{t+1}$ equally, hence permanently

  - with β endogenous, the reduction in wealth reduces consumption which raises λ_t and $\beta(c_t, h_t)$

    * increase in $E_t \lambda_{t+1}$ is less than increase in λ_t

* eventually λ_t returns to its steady-state value

- c_t

$$\lambda_t = U_c(c_t, h_t) - \eta_t \beta_c(c_t, h_t)$$

  – Marginal utility of wealth (λ_t) equals marginal utility of consumption $[U_c(c_t, h_t)]$ less the marginal value of the reduction in the discount factor (positive term)

  – Unit decline in discount factor in turn reduces utility in period t by η_t

- h_t

$$U_h(c_t, h_t) + \lambda_t A_t F_h(k_t, h_t) - \eta_t \beta_h(c_t, h_t) = 0$$

$$-[U_h(c_t, h_t) - \eta_t \beta_h(c_t, h_t)] = A_t F_h(k_t, h_t)[U_c(c_t, h_t) - \eta_t \beta_c(c_t, h_t)]$$

  - marginal disutility of hours includes effect of increase in hours in increasing the discount factor
  - adjusted marginal disutility of hours equals the wage times the adjusted marginal utility of consumption

- θ_{t+1}

$$\eta_t = -E_t U\left(c_{t+1}, h_{t+1}\right) + E_t \eta_{t+1} \beta\left(c_{t+1}, h_{t+1}\right)$$

  - Solving forward

$$\eta_t = -E_t \sum_{j=1}^{\infty} \frac{\theta_{t+j}}{\theta_{t+1}} U\left(c_{t+j}, h_{t+j}\right)$$

  - η_t is the negative of the present discounted value of utility

- k_{t+1}

$$\begin{aligned} & \lambda_t \left[1 + \Phi'\left(k_{t+1} - k_t\right)\right] \\ = \ & \beta\left(c_t, h_t\right) E_t \lambda_{t+1} \left[A_{t+1} F_k\left(k_{t+1}, h_{t+1}\right) + 1 - \delta + \Phi'\left(k_{t+2} - k_{t+1}\right)\right] \end{aligned}$$

  - cost of one additional unit of capital includes adjustment costs
  - benefits are discounted utility value of capital's marginal product plus its undepreciated value plus saved adjustment cost going forward since capital it higher

1.5 Assumptions

- Perfect capital mobility such that interest rate in small open economy equals fixed world rate

$$r_t = r$$

- Productivity is stationary and AR(1)

$$\ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1} \quad \rho \in (-1, 1) \quad \epsilon \sim N\left(0, \sigma_\epsilon^2\right)$$

- Functional forms

$$U(c, h) = \frac{\left[c - \omega^{-1} h^{\omega}\right]^{1-\gamma} - 1}{1 - \gamma}$$

$$\beta(c,h) = \left[1 + c - \omega^{-1}h^{\omega}\right]^{-\psi_1}$$

$$F(k,h) = k^{\alpha}h^{1-\alpha}$$

$$\Phi(x) = \frac{\phi}{2}x^2 \quad \phi > 0$$

- Combining FO conditions on c_t and h_t, functional forms make labor supply independent of consumption (ratios of marginal utilities depend only on labor)

$$A_t(1-\alpha)k_t^{\alpha}h_t^{-\alpha} = h_t^{\omega-1}$$

  - lhs is wage at which firm is willing to hire labor, hence labor demand and rhs is wage as which household willing to supply hours, hence labor supply

2 Calibration to Canadian Economy (Annual)

- $\omega = 1.455$ implies a high labor supply elasticity of $\frac{1}{\omega-1} = 2.2$
  - Totally differentiate second equation below with respect to wage and labor supply

$$A(1-\alpha)k^{\alpha}h^{-\alpha} = h^{\omega-1} = w_t$$

$$dw = (\omega-1)h^{\omega-2}dh$$

$$\frac{dh}{dw} = \frac{1}{\omega-1}h^{\omega-2}$$

$$\frac{wdh}{hdw} = \frac{1}{\omega-1}$$

- $\delta = 0.1$ implies that capital depreciates at 10% per year

- $\alpha = .32$ implies a capital income share of 32%

- $r = .04$ is the average real rate of return of broad measures of the stock market in developed countries post WWII

- ψ_1 is the elasticity of the discount factor with respect to the composite $1 + c - \omega^{-1}h^{\omega}$

  - match average Canadian trade-balance-to-GDP ratio of 2%

  - steady state value of FO condition on bonds yields

$$\beta(c, h)(1 + r) = 1$$

– steady state value of FO condition on capital with $A = 1$ yields

$$r + \delta = \alpha \left(\frac{h}{k}\right)^{1-\alpha}$$

$$\frac{k}{h} = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}}$$

implying that the steady-state capital labor ratio is independent of ψ_1

– steady state value of hours is also independent of ψ_1

$$h = \left[(1-\alpha)\left(\frac{k}{h}\right)^{\alpha}\right]^{\frac{1}{\omega-1}}$$

– steady-state values of capital, output, and investment $(i = \delta k)$ will all be independent of ψ_1

– steady-state value of FO condition on bonds

$$\beta(c,h)(1+r) = \left[1 + c - \omega^{-1}h\right]^{-\psi_1}(1+r) = 1$$

* use resource constraint

$$c = y - i - tb$$

to substitute for c

$$\left[1 + y - i - tb - \omega^{-1}h\right]^{-\psi_1}(1+r) = 1$$

* solve for $\frac{tb}{y}$

$$\frac{tb}{y} = 1 - \frac{i}{y} - \frac{(1+r)^{\frac{1}{\psi_1}} + \omega^{-1}h - 1}{y}$$

- since hours, capital, investment, and output are independent of ψ_1, can calibrate ψ_1 to match $\frac{tb}{y}$

- find ψ_1 is increasing in $\frac{tb}{y}$

- dual role of ψ_1

  * determines steady-state trade balance/output ratio

  * governs speed of convergence to steady state

  * might want to separate these allowing speed of convergence to be small enough as to not significantly affect business cycle dynamics

  * add a parameter and respecify $\beta(c_t, h_t)$

$$\beta(c_t, h_t) = \bar{\beta}\left[1 + (c_t - c) - \omega^{-1}\left(h_t^{\omega} - h^{\omega}\right)\right]^{-\psi_1}$$

where c and h are steady-state values

- $\phi = 0.028$ matches standard deviation of investment of 9.8

- $\sigma\epsilon = 0.0129$ matches standard deviation of output of 2.8

- $\rho = 0.42$ matches serial correlation of output of 0.61

3 Equilibrium

3.1 Definition

A competitive equilibrium is a set of processes $\{d_t, c_t, h_t, y_t, i_t, k_{t+1}, \eta_t, \lambda_t, A_t\}$ satisfying the resource constraint and the first order conditions, given the fixed world interest rate, r, initial conditions A_0, d_{-1}, and k_0 and the exogenous process $\{\epsilon_t\}$

3.2 Approximation

- Solutions where endogenous variables fluctuate in small neighborhood around steady state

  – Debt will be bounded implying

$$\lim_{j \to \infty} E_t d_{t+j} \left(\frac{1}{1+r} \right)^j = 0$$

  – Write system as

$$E_t f \left(x_{t+1}, x_t \right) = 0$$

  – Cannot solve non-linear system

- Linearize about steady state

– Express most variables as percent deviations about steady state

$$\hat{w}_t \equiv \log(w_t/w) \approx \frac{w_t - w}{w}$$

– For variables that can take on negative values, like trade balance, or variables already expressed as percent, like interest rates, just use first difference

$$\hat{w}_t \equiv w_t - w$$

– Linearized system is expressed as

$$A\hat{x}_{t+1} = B\hat{x}_t$$

where A and B are conformable square matrices made up of known coefficients in the calibrated linearized model

- Ten variables in linearized system
  - state variables
    - variables whose t values are predetermined (determined before t)
    - variables whose values are exogenous
    - include $\hat{k}_t, \hat{d}_{t-1}, \hat{A}_t$, and $\hat{r}$
  - co-state variables
    - endogenous variables have values not predetermined in period t

- Initial conditions
  - Three known initial conditions $\hat{k}_0, \hat{d}_{-1}, \hat{A}_0$
  - Determine other initial conditions to satisfy boundedness condition

$$\lim_{j \to \infty} \left[E_t \hat{x}_{t+j} \right] = 0$$

4 Model Performance

4.1 Successes

- Model is calibrated to match some moments and does well here by construction
  - volatility of output
  - volatility of investment,
  - serial correlation of output

- Volatility rankings
  - investment volatility greater than output volatility
  - consumption volatility less than output volatility
- trade balance is countercyclical
  - productivity shock increases investment more than it reduces savings due to consumption smoothing
  - result due to parameters ϕ, governing cost of investment, and ρ, persistence of productivity shock and these values were calibrated independent of trade balance performance

4.2 Failures

- too little countercyclicality in trade balance
  - need investment and/or consumption to increase more in response to positive productivity shock

- correlation between
  - consumption and output too high
  - between hours and output is too high at unity
    - due to functional form for the period utility index

* condition for equilibrium hours

$$A_t\,(1-\alpha)\,k_t^{\alpha}h_t^{-\alpha}=h_t^{\omega-1}$$

$$(1-\alpha)\,y_t=h_t^{\omega}$$

* log-linearized version is

$$\hat{y}_t=\omega\hat{h}_t$$

* implying a correlation of unity

5 Impulse responses

- Output, investment, hours, and consumption all respond positively to a productivity innovation

- The trade balance/GDP and the current account/GDP both respond negatively

5.1 Countercyclicality of the trade balance

- Investment must respond strongly enough
  - requires that adjustment cost not be too high
  - persistence of productivity must be high enough

- Consumption must respond strongly enough
  - requires persistence relatively high

6 Other Ways to Impose Stationarity

6.1 Debt-elastic interest rate

- Assume interest rate paid by small open economy is increasing in its aggregate external debt

$$r_t = r + p\left(\tilde{d}_t\right)$$

  - Introduces a risk-premium on debt

- Assume agent does not take into account the effect of his change in debt on the country risk premium implying that the Euler equation is

$$\lambda_t = \beta\left(1 + r_t\right)\lambda_{t+1}$$

- Dynamics of the model: begin away from steady state and show that end up in steady state
  - Assume $\beta(1+r_t) > 1$
  - Agent are saving, wealth and consumption are rising and marginal utility (λ) is falling
  - Increase in wealth implies a reduction in foreign debt and a reduction in r_t, returning system to steady state
- Can calibrate the p parameter to be small enough that it yields stationarity without affecting business cycle dynamics

6.2 Portfolio Adjustment Costs

- Assume there is a cost to holding debt in quantity different from its long-run equilibrium value

$$d_t = (1 + r_{t-1})\, d_{t-1} - y_t + c_t + k_{t+1} - (1 - \delta)\, k_t + \Phi\, (k_{t+1} - k_t) + \frac{\psi_3}{2} \left(d_t - \bar{d}\right)^2$$

reducing the marginal utility of debt due to the adjustment cost

- Euler equation becomes

$$\lambda_t \left[1 - \psi_3 \left(d_t - \bar{d}\right)\right] = \beta\, (1 + r)\, E_t \lambda_{t+1}$$

- Dynamics of the model: shock model away from steady state and show that end up in steady state

– Assume d_t increases from steady state value such that $d_t > \bar{d}$

$$\left[1 - \psi_3 \left(d_t - \bar{d}\right)\right] = \beta \left(1 + r\right) \frac{E_t \lambda_{t+1}}{\lambda_t}$$

– lhs decreases below unity requiring

$$\frac{E_t \lambda_{t+1}}{\lambda_t} < 1$$

– the increase in debt is a reduction in wealth such that consumption falls and marginal utility rises

– adjustment costs require that $E_t \lambda_{t+1}$ rise by less than λ_t rises, requiring that current consumption fall by more than future consumption

– agent begins saving to replace wealth lost by shock returning system to steady state

- Can calibrate ψ_3 to be small enough that it yields stationarity without affecting business cycle dynamics much

6.3 Precautionary Savings

- Euler equation with log utility

$$\frac{1}{c_t} = \beta (1+r) E_t \frac{1}{c_{t+1}} > \beta (1+r) \frac{1}{E_t c_{t+1}}$$

- With $\beta (1+r) = 1$

$$E_t c_{t+1} > c_t$$

  - Such that consumption is rising implying that wealth is growing
  - Increase in wealth reduces magnitude of precautionary motive implying wealth is rising at falling rate
  - Eventually reach a steady state

  - However, since world cannot have steady state at $\beta(1+r)=1$, really need to consider $\beta(1+r)<1$ to reduce world savings

- Problem

  - Precautionary saving requires a positive third derivative of utility

  - Cannot use in a linearized model because linearization eliminates the third derivative

7 Business Cycles in Emerging Markets (Aguiar and Gopinath 2007)

7.1 Model

- Utility per capita

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[C_t^{\gamma} (1 - h_t)^{1-\gamma}\right]^{1-\sigma} - 1}{1 - \sigma}$$

- Budget constraint per capita

$$\frac{D_{t+1}}{R_t} = D_t + C_t + K_{t+1} - (1 - \delta) K_t + \frac{\phi}{2} \left(\frac{K_{t+1}}{K_t} - g \right)^2 K_t - A_t K_t^{\alpha} (X_t h_t)^{1-\alpha}$$

- First order conditions for household letting Λ_t denote multiplier

  - C_t

$$\left[C_t^{\gamma}(1-h_t)^{1-\gamma}\right]^{-\sigma}\gamma C_t^{\gamma-1}(1-h_t)^{1-\gamma}-\Lambda_t=0$$

  - h_t

$$-\left[C_t^{\gamma}(1-h_t)^{1-\gamma}\right]^{-\sigma}(1-\gamma)C_t^{\gamma}(1-h_t)^{-\gamma}+\Lambda_t(1-\alpha)A_tK_t^{\alpha}X_t^{1-\alpha}h_t^{-\alpha}$$

  - D_{t+1}

$$\Lambda_t\frac{1}{R_t}-E_t\Lambda_{t+1}\beta=0$$

- K_{t+1}

$$\Lambda_t \left(1 + \phi \left(\frac{K_{t+1}}{K_t} - g\right)\right)$$
$$= \beta E_t \Lambda_{t+1} [1 - \delta + \alpha A_{t+t} K_{t+1}^{\alpha-1} \left(X_{t+1} h_{t+1}\right)^{1-\alpha}$$
$$+\phi \left(\frac{K_{t+2}}{K_{t+1}} - g\right) \frac{K_{t+2}}{K_{t+1}} - \frac{\phi}{2} \left(\frac{K_{t+2}}{K_{t+1}} - g\right)^2]$$

- Stationary productivity shock

$$\ln A_t = \rho_a \ln A_{t-1} + \sigma_a \epsilon_t^a \quad 0 < \rho_a < 1$$

- Non-Stationary productivity shock

  - Define

$$g_t = \frac{X_t}{X_{t-1}}$$

  - Growth is stationary

$$\ln (g_t - g) = \rho_g \ln (g_{t-1} - g) + \sigma_g \epsilon_t^g \quad 0 < \rho_g < 1 \quad g > 1$$

  - Unit root in productivity gives other variables unit root, so equilibrium is not stationary

- Find a way to transform the model to make transformed variables stationary

7.2 Transformation to make the model stationary

- Divide trending variables by X_{t-1} and define resulting variables, like $\frac{C_t}{X_{t-1}} \equiv c_t$

- In FO condition on C_t, expressing in terms of c_t requires multiplying equation by $X_{t-1}^{1+\gamma(\sigma-1)}$

  – Therefore, define

$$\lambda_t = \Lambda_t X_{t-1}^{1+\gamma(\sigma-1)}$$

- Detrended FO conditions

- c_t

$$\left[c_t^{\gamma}(1-h_t)^{1-\gamma}\right]^{-\sigma}\gamma c_t^{\gamma-1}(1-h_t)^{1-\gamma}-\lambda_t=0$$

- h_t multiply equation by $X_{t-1}^{\gamma(\sigma-1)}$

$$\left[c_t^{\gamma}(1-h_t)^{1-\gamma}\right]^{-\sigma}(1-\gamma)\,c_t^{\gamma}(1-h_t)^{-\gamma}=\lambda_t(1-\alpha)\,A_t k_t^{\alpha}g_t^{1-\alpha}h_t^{-\alpha}$$

- Using the two equations together

$$\frac{(1-\gamma)\,c_t}{\gamma(1-h_t)}=(1-\alpha)\,A_t g_t\left(\frac{k_t}{g_t h_t}\right)^{\alpha}$$

- D_{t+1} multiply equation by $X_{t-1}^{1+\gamma(\sigma-1)}$

$$\lambda_t\frac{1}{R_t}-\beta E_t\Lambda_{t+1}X_{t-1}^{1+\gamma(\sigma-1)}=0$$

$$\lambda_t \frac{1}{R_t} - \beta E_t \lambda_{t+1} g_t^{-1-\gamma(\sigma-1)} = 0$$

– K_{t+1}

$$\frac{K_{t+2}}{K_{t+1}} = \frac{\frac{K_{t+2}}{X_t}\frac{X_{t+1}}{X_{t+1}}}{\frac{K_{t+1}}{X_t}} = \frac{g_{t+1}k_{t+2}}{k_{t+1}}$$

$$\lambda_t \left(1 + \phi\left(\frac{g_t k_{t+1}}{k_t} - g\right)\right)$$
$$= \beta g_t^{-1-\gamma(\sigma-1)} E_t \lambda_{t+1}[1 - \delta + \alpha A_{t+t}\left(\frac{k_{t+1}}{g_{t+1}h_{t+1}}\right)^{\alpha-1}$$
$$+\phi\left(\frac{g_{t+1}k_{t+2}}{k_{t+1}} - g\right)\frac{g_{t+1}k_{t+2}}{k_{t+1}} - \frac{\phi}{2}\left(\frac{g_{t+1}k_{t+2}}{k_{t+1}} - g\right)^2]$$

– Linearized system has the property that a transitory shock will create a permanent increase in consumption if R_t is exogenous

- Risk premium on interest rate increasing in debt above some minimum level, $\bar{d}$

$$R_t = R^* + \psi \left[e^{d_{t+1} - \bar{d}} - 1 \right]$$

7.3 Equilibrium

7.4 Properties of equilibrium

- Stationary in detrended variables
- Actual variables inherit the unit root in X_t since

$$C_t = c_t X_{t-1}$$

- Stationary distribution of wealth around its mean due to assumption about risk-premium on interest rate

7.5 Calibration

- Calibrate $\beta = 0.98$, $\gamma = 0.36$, $\psi = 0.001$, $\alpha = 0.68$, $\sigma = 2$, $\delta = 0.05$

- Estimate parameters for stochastic processes and adjustment cost ϕ at quarterly horizon using Mexico 1980Q1 to 2003Q1.
  - $\sigma_g = 0.0213 \quad \rho_g = 0.00 \quad g = 1.0066$
  - $\sigma_a = 0.0053 \quad \rho_a = 0.95$
  - $\phi = 1.37$

- What fraction of variance of Solow residual is determined by the permanent shock?

– Detrended output

$$A_t k_t^{\alpha} \left(g_t h_t\right)^{1-\alpha}$$

– Solow Residual

$$SR_t = A_t g_t^{1-\alpha}$$

– Fraction of variance of Solow residual accounted for by non-stationary shock is 88%

– Compares to 40% for Canada

– Implies that importance of non-stationary shock is major difference in rich and emerging market countries

- Potential problem

– Need long time series to confidently identify a data series as non-stationary

7.6 Compare moments to data

- Since data is non-stationary, need to make it stationary to compute moments
  - Unconditional mean and variance do not exist in non-stationary data
  - Use HP filter
  - Theoretically should estimate the non-stationary trend and deflate by that as in model
- Consumption is more volatile than output
  - Importance of non-stationary output

- Trade balance is countercyclical
- Other matches also good