Term Structure

Consider a simplification of the model of Vasicek [1] of the term structure of interest rates.

The short-term, risk-free interest rate r follows a random walk,

$$\mathrm{d}r = \rho\,\mathrm{d}z.$$

Let $P(\tau, r)$ denote the price of a risk-free pure discount bond worth one dollar at its maturity in τ years. Of course $P(0, r) = 1$. We wish to solve for the equilibrium price.

Yield to Maturity

Let $R(\tau, r)$ denote the yield to maturity on the τ-year bond. By definition,

$$P(\tau, r) = \mathrm{e}^{-\tau R(\tau, r)},$$

so

$$R(\tau, r) = -\frac{1}{\tau} \ln P(\tau, r).$$

Expectations Theory of the Term Structure

The standard model of the term structure is the expectations theory, which argues that the long-term interest rate is the average of the current and expected future short-term interest rates.

Here the expected future short-term rate is just the current short-term rate, so

$$R(\tau, r) = r$$

according to the expectations theory. Hence

$$P(\tau, r) = \mathrm{e}^{-r\tau}.$$

Return

The price of a bond at time t maturing at time T is $P(T - t, r)$. The return on the bond is the price change $\mathrm{d}P/P$.

By Itô's formula,

$$\begin{aligned}
\mathrm{d}P &= -\frac{\partial P}{\partial \tau}\mathrm{d}t + \frac{\partial P}{\partial r}\mathrm{d}r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\mathrm{d}r)^2 \ \text{(τ falls as t rises)} \\
&= -\frac{\partial P}{\partial \tau}\mathrm{d}t + \frac{\partial P}{\partial r}\rho\,\mathrm{d}z + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\rho\,\mathrm{d}z)^2 \\
&= \left(-\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \rho\frac{\partial P}{\partial r}\,\mathrm{d}z.
\end{aligned}$$

Market Equilibrium

For market equilibrium, assume that all bonds must have expected rate of return r:

$$r\,\mathrm{d}t = \mathrm{E}_t\left(\frac{\mathrm{d}P}{P}\right) = \frac{1}{P}\left(-\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t.$$

Term-Structure Equation

We wish to solve the term-structure equation

$$rP = -\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}, \tag{1}$$

subject to the boundary condition $P(0, r) = 1$.

Constant Interest Rate

The special case $\rho = 0$ implies a constant interest rate. The term-structure equation is then

$$rP = -\frac{\partial P}{\partial \tau},$$

with solution

$$P(\tau, r) = \mathrm{e}^{-r\tau}.$$

The yield to maturity is

$$R(\tau, r) = r,$$

in agreement with the expectations theory.

General Solution

The general solution is

$$P(\tau, r) = \mathrm{e}^{-r\tau + \frac{1}{6}\rho^2\tau^3},$$

which one verifies by substituting into the term-structure equation.

Return

The return is

$$\begin{aligned}\frac{\mathrm{d}P}{P} &= \left(-\frac{1}{P}\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2 \frac{1}{P}\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \rho \frac{1}{P}\frac{\partial P}{\partial r}\,\mathrm{d}z \\ &= \left[\left(r - \frac{1}{2}\rho^2\tau^2\right) + \frac{1}{2}\rho^2\tau^2\right]\mathrm{d}t - \rho\tau\,\mathrm{d}z \\ &= r\,\mathrm{d}t - \tau\,\mathrm{d}r.\end{aligned}$$

An increase in r reduces P, and the standard deviation of the return is proportional to the term to maturity.

Yield to Maturity

The yield to maturity is

$$R(\tau, r) = -\frac{1}{\tau}\ln P(\tau, r) = r - \frac{1}{6}\rho^2\tau^2.$$

The yield to maturity at time t on a bond maturing at time T is $R(T-t, r)$, which follows the stochastic differential equation

$$\begin{aligned}\mathrm{d}R &= -\frac{\partial R}{\partial \tau}\mathrm{d}t + \frac{\partial R}{\partial r}\mathrm{d}r + \frac{1}{2}\frac{\partial^2 R}{\partial r^2}(\mathrm{d}r)^2 \\ &= -\left[-\frac{1}{3}\rho^2(T-t)\right]\mathrm{d}t + 1\,\mathrm{d}r + \frac{1}{2}0\,(\mathrm{d}r)^2 \\ &= \frac{1}{3}\rho^2(T-t)\,\mathrm{d}t + \mathrm{d}r.\end{aligned}$$

Risk Premium

Alternatively, one might allow the possibility of a risk premium. The stochastic differential for the price takes the form

$$\frac{\mathrm{d}P}{P} = m(\tau, r)\,\mathrm{d}t + s(\tau, r)\,\mathrm{d}z.$$

The returns for the different bonds are perfectly correlated, since each involves the same instantaneous error $\mathrm{d}z$.

No Arbitrage

Consequently there will be an arbitrage opportunity unless the risk premium is proportional to the standard deviation:

$$m(\tau,r) - r \propto s(\tau,r).$$

Let q denote the proportionality factor.

Term-Structure Equation

The term structure equation is

$$m(\tau,r) - r = qs(\tau,r),$$

which takes the form

$$\left(-\frac{1}{P}\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2 \frac{1}{P}\frac{\partial^2 P}{\partial r^2}\right) - r = q\left(-\rho\frac{1}{P}\frac{\partial P}{\partial r}\right).$$

Hence the term-structure equation (1) changes to

$$rP = -\frac{\partial P}{\partial \tau} + q\rho\frac{\partial P}{\partial r} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}.$$

To express q as a function of r would be a natural model.

Constant Risk Premium

For constant q, the bond price is

$$P(\tau,r) = \mathrm{e}^{-r\tau - \frac{1}{2}q\rho\tau^2 + \frac{1}{6}\rho^2\tau^3}.$$

The yield to maturity is

$$R(\tau,r) = -\frac{1}{\tau}\ln P(\tau,r) = r + q\rho\tau - \frac{1}{6}\rho^2\tau^2.$$

Return

The return is

$$\begin{aligned}\frac{\mathrm{d}P}{P} &= \left(-\frac{1}{P}\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2\frac{1}{P}\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \rho\frac{1}{P}\frac{\partial P}{\partial r}\,\mathrm{d}z \\ &= \left[\left(r + q\rho\tau - \frac{1}{2}\rho^2\tau^2\right) + \frac{1}{2}\rho^2\tau^2\right]\mathrm{d}t - \rho\tau\,\mathrm{d}z \\ &= (r + q\rho\tau)\,\mathrm{d}t - \tau\,\mathrm{d}r.\end{aligned}$$

References

[1] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, November 1977. HB1J69X.