### Two Equivalent Conditions

The traditional theory of present value puts forward two equivalent conditions for asset-market equilibrium:

**Rate of Return**  The expected rate of return on an asset equals the market interest rate;

**Present Value**  The asset price equals the present value of expected future payments.

We explain these two conditions and show that they are equivalent—either condition implies the other.

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<td><strong>Market Interest Rate</strong></td>
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<td>The rate-of-return condition says just that all assets share a common expected rate of return. The market interest rate refers to the expected rate of return common to all assets.</td>
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<td>We assume that the market interest rate $R &gt; 0$ is constant.</td>
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<td><strong>Notation</strong></td>
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<td>Consider an asset with payment $S_t$ at time $t$. For a stock, the payment would be the dividend. For a bond, the payment would be interest or principal.</td>
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<td>Let $P_t$ denote the asset price at time $t$.</td>
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<td><strong>Expected Rate of Return</strong></td>
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<td><strong>Definition 1 (Return)</strong>  The return is the profit divided by the amount invested.</td>
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<tr>
<td><strong>Definition 2 (Expected Rate of Return)</strong>  The expected rate of return is the expected return divided by the length of the time period.</td>
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<td><strong>Disequilibrium</strong></td>
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<td>If the expected rate of return were greater than the market interest rate, the security would be seen as a “good buy.” Investors would like to buy the security; those holding the security would not want to sell it. Demand would exceed supply. The reverse inequality would lead to excess supply.</td>
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<td><strong>Definition 3 (Present Value)</strong>  The present value of a payment to be received in the future is the dollars attainable now by borrowing against the future payment.</td>
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<td><strong>Definition 4 (Discount Factor)</strong>  The present value is the future payment multiplied by the discount factor.</td>
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**Discrete Time**

With compound interest, a dollar borrowed at time 0 will require a repayment of $(1 + R)^t$ at time $t$, the principal plus interest.

**Theorem 5 (Present Value)** The present value at time 0 of $S_t$ dollars at time $t$ is

$$\frac{S_t}{(1 + R)^t}$$

dollars. The discount factor is $1 / (1 + R)^t$.

**Simple Example of Equivalence**

Consider an asset paying $P_1$ at time 1 and paying nothing at other times. Suppose that the interest rate is $R$. What would be a fair price $P_0$ to pay for the asset at time 0?

**Rate-of-Return Condition**

Using the rate-of-return condition, what would be a fair price $P_0$ to pay for the asset at time 0? Setting the rate of return equal to the market interest rate gives

$$\frac{P_1 - P_0}{P_0} = R;$$

the profit is the capital gain. Solving for the price gives

$$P_0 = \frac{P_1}{1 + R}. \quad (1)$$

**Present-Value Condition**

For this asset, the present-value condition says that the market price equals the present value of expected payments,

$$P_0 = \frac{P_1}{1 + R}.$$

But this condition is identical to (1), obtained from the rate-of-return condition.

**Continuous Time**

Here $S_t$ is the payment flow.

For an investment from time $t$ to time $t + dt$, the profit is the payment $S_t dt$ plus the capital gain $dP_t$. The return during the period is the profit divided by the beginning-of-period price,

$$\frac{S_t dt + dP_t}{P_t}.$$
**Rate-of-Return Equilibrium Condition**

**Condition 6 (Rate-of-Return Equilibrium Condition)** The expected rate of return equals the market interest rate,

\[
\frac{S_t}{P_t} dt + E_t \left( \frac{dP_t}{Pt} \right) = R_t dt. \tag{2}
\]

**Theorem 7 (Present Value)** The present value at time 0 of one dollar at time \( t \) is \( e^{-R_t} \) dollars, and the discount factor is \( e^{-R_t} \).

The present-value condition for asset-market equilibrium asserts that the asset price equals the present value of expected payments,

\[
P_0 = \int_0^\infty e^{-R_t} E_0 (S_t) \, dt. \tag{3}
\]

**Perpetual Bond**

Consider a perpetual bond, a bond paying one-dollar interest per period in perpetuity; the principal is never repaid. If the interest rate is \( R \), what is a fair price for the bond?

A fair price is

\[
\frac{1}{R}, \tag{4}
\]

for the bond then has rate of return \( R \). For example, if \( R = 0.10 \), then the bond should sell for $10, so it will return 10%. A lower price would give a higher yield, and a higher price would give a lower yield.

**Present Value**

Consider an investment worth \( P_t \) at time \( t \). If the investment earns the market interest rate \( R \), then with continuous compounding its value follows the differential equation

\[ dP_t = RP_t \, dt, \]

with the solution

\[ P_t = P_0 e^{Rt}. \]

One dollar invested at time 0 is worth \( e^{Rt} \) dollars at time \( t \). Conversely, if one borrows \( e^{-Rt} \) dollars at time 0, with interest one will owe 1 dollar at time \( t \).

**Equivalence**

The rate-of-return condition (2) is equivalent to the present-value condition (3).

We first demonstrate the equivalence in several examples and then give a general proof.

**Present Value**

To show the equivalence between the two equilibrium conditions, we must show that the present value of the payments is (4).
The present value is
\[
\int_0^\infty e^{-Rt} S_t \, dt = \int_0^\infty e^{-Rt} \, 1 \, dt
\]
\[
= - \frac{1}{R} \left[ e^{-Rt} \right]_0^\infty
\]
\[
= \frac{1}{R} e^{-R0} - \frac{1}{R} \lim_{t \to \infty} e^{-Rt}
\]
\[
= \frac{1}{R}
\]
in accord with the expected rate-of-return reasoning.

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Stock

Next we analyze a related but more complex example. Consider a stock paying dividend \(D_t\), and the dividend grows at the constant rate \(G\). Here
\[S_t = D_t = D_0 e^{Gt}.\]

What is a fair price \(P_0\) for the stock at time 0?

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Rate-of-Return for the Stock

The rate of return is the dividend yield plus the rate of capital gain. The dividend yield is \(D_t/P_t\). Since the dividend grows each period at rate \(G\), intuitively the stock price should also grow at this rate:
\[
\frac{dP_t}{P_t} = G \, dt;
\]
the rate of capital gain is \(G\). The return is therefore
\[
\frac{D_t \, dt + P_t \, G \, dt}{P_t}.\]

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Present Value for the Stock

This price equals the present value at time 0 of the payments:
\[
\int_0^\infty e^{-Rt} S_t \, dt = \int_0^\infty e^{-Rt} \left( D_0 e^{Gt} \right) \, dt
\]
\[
= - \frac{D_0}{R-G} \left[ e^{-(R-G)t} \right]_0^\infty
\]
\[
= \frac{D_0}{R-G} e^{-(R-G)0} - \frac{D_0}{R-G} \lim_{t \to \infty} e^{-(R-G)t}
\]
\[
= \frac{D_0}{R-G},
\]
in agreement with (5).

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This result requires \(R > G\). Without this condition, the dividend rises faster than the discount factor falls, and the present value is infinite.

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Financial Economics  
Present Value  

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**General Equivalence**

In general, the two conditions for equilibrium are equivalent. If the price equals the present value at every moment, then the rate of return equals the market interest rate at every moment; and vice versa. We prove the equivalence.

**Theorem 10** If the rate-of-return equilibrium condition (2) holds at every moment, then

\[ P_t = \int_{t}^{\infty} e^{-R(\tau-t)} E_t(S_\tau) \, d\tau + \lim_{\tau \to \infty} \left[ e^{-R(\tau-t)} E_t(P_\tau) \right]. \]

**Condition 8 (Present-Value Equilibrium Condition)** The asset price equals the present value of expected payments,

\[ P_t = \int_{t}^{\infty} e^{-R(\tau-t)} E_t(S_\tau) \, d\tau. \]  

**Theorem 9** If the present-value equilibrium condition (6) holds at every moment, then the rate-of-return equilibrium condition (2) holds at every moment.

**Condition 11 (No-Bubble)** As the future time goes to infinity, the present value of the expected future price goes to zero:

\[ \lim_{\tau \to \infty} \left[ e^{-R(\tau-t)} E_t(P_\tau) \right] = 0. \]  

**Corollary 12** If the rate-of-return equilibrium condition (2) holds at every moment, and the no bubble condition (7) holds, then the present-value equilibrium condition (6) holds at every moment.

**Bubble**

A *bubble* refers to a situation in which the asset price is not set by expected future payments but instead is driven by the expectation of high capital gains. People pay a high price for the asset because its price is rising and they hope for further increases. The prospects for future payments are unimportant, as the asset owner hopes to sell the asset to someone else at a high price.

The terminology comes from the soap bubbles blown by children. The bubbles have nothing inside, and soon they pop. An asset bubble pops at some point, and the price falls.

**No Uncertainty**

For simplicity, first we assume no uncertainty.

First, assume (6): at every moment the price equals the present value.

We use the formula for the differentiation of an integral:

\[ \frac{d}{dt} \left[ \int_{a(t)}^{b(t)} f(\tau,t) \, d\tau \right] = b'(t) f[b(t),t] - a'(t) f[a(t),t] + \int_{a(t)}^{b(t)} \frac{\partial f(\tau,t)}{\partial t} \, d\tau. \]

In the integral, \( t \) appears three times, so the derivative has three terms.
Differentiating (6) obtains
\[
\frac{dP_t}{dt} = d \left[ \int_t^\infty e^{-R(t-\tau)} S_\tau \ d\tau \right] = (-1) e^{-R(t-\tau)} S_\tau \ dt + R \left[ \int_t^\infty e^{-R(t-\tau)} S_\tau \ d\tau \right] \ dt \\
= -S_\tau \ dt + RP_t \ dt.
\]
Rearranging obtains the rate-of-return condition (2).

Multiply by the discount factor and integrate,
\[
\int_t^\infty e^{-R(t-\tau)} S_\tau \ d\tau = \int_t^\infty e^{-R(t-\tau)} (RP_t \ d\tau - dP_t) \]
\[
= \int_t^\infty d \left[ -e^{-R(t-\tau)} P_\tau \right] \]
\[
= -e^{-R(t-\tau)} P_t \bigg|_t^\infty \\
= P_t - \lim_{\tau \to \infty} e^{-R(t-\tau)} P_\tau.
\]
We have proved (10) for the case of no uncertainty.

Take the expected value:
\[
E_t \left( \frac{dP_t}{dt} \right) = (-S_\tau + RP_t) dt \\
+ \int_t^\infty e^{-R(t-\tau)} E_t \left[ E_{\tau+dt} (S_\tau) - E_t (S_\tau) \right] \ d\tau \\
= (-S_t + RP_t) dt.
\]
Compared with the certainty case there is an extra term, but this extra term is zero:
\[
E_t \left[ E_{\tau+dt} (S_\tau) - E_t (S_\tau) \right] = 0.
\]
The change in the expected value is an innovation, and the expected value of an innovation is zero. Rearranging obtains (2).

Reverse Implication

Conversely, suppose that the return equals the market return at every moment, and we work backwards by integrating.

Write (2) as
\[
S_t \ dt = RP_t \ dt - dP_t.
\]

Uncertainty

We extend the proof to uncertainty.

First, assume (6): at every moment the price equals the present value. Then
\[
\frac{dP_t}{dt} = d \left[ \int_t^\infty e^{-R(t-\tau)} E_t (S_\tau) \ d\tau \right] = -S_t \ dt + \int_t^\infty e^{-R(t-\tau)} \left[ E_{\tau+dt} (S_\tau) - E_t (S_\tau) \right] \ d\tau.
\]

Alternative Derivation

\[
P_t = \int_t^\infty e^{-R(t-\tau)} E_t (S_\tau) \ d\tau \\
= \int_t^{t+dt} e^{-R(t-\tau)} E_t (S_\tau) \ d\tau + \int_{t+dt}^\infty e^{-R(t-\tau)} E_t (S_\tau) \ d\tau \\
= e^{-R(t-\tau)} E_t (S_\tau) \ dt + e^{-R(t-(\tau+dt))} E_{\tau+dt} (S_\tau) \ d\tau \\
= \int_{t+dt}^\infty e^{-R(t-(\tau+dt))} E_{\tau+dt} (S_\tau) \ d\tau,
\]
since
\[
E_t \left[ E_{\tau+dt} (S_\tau) \right] = E_t (S_\tau).
\]
Since the expression in braces is $P_{t+dt}$,

$$P_t = S_t dt + e^{-R dt} E_t (P_{t+dt})$$

$$= S_t dt + (1 - R dt) [P_t + E_t (dP_t)]$$

$$= S_t dt + (1 - R dt) P_t + E_t (dP_t)$$

as $dt dP_t = 0$,

which simplifies to (2).

The expected value of the left-hand side is

$$E_t \left[ \int_t^\infty e^{-R(\tau-t)} (S_\tau d\tau - s_\tau dz_\tau) \right] = \int_t^\infty e^{-R(\tau-t)} E_t (S_\tau) d\tau,$$

as

$$E_t (s_\tau dz_\tau) = 0.$$

The right-hand side is

$$\int_t^\infty e^{-R(\tau-t)} (RP_\tau d\tau - dP_\tau) = \int_t^\infty d \left[ -e^{-R(\tau-t)} P_\tau \right]$$

$$= -e^{-R(\tau-t)} P_\tau \right]_t^\infty$$

$$= P_t - \lim_{\tau \to \infty} e^{-R(\tau-t)} P_\tau.$$

Taking the expected value,

$$E_t \left[ \int_t^\infty e^{-R(\tau-t)} (RP_\tau d\tau - dP_\tau) \right] = P_t - \lim_{\tau \to \infty} e^{-R(\tau-t)} E_t (P_\tau).$$

Hence theorem (10) follows.

### Reverse Implication

By (2), then

$$dP_t = (-S_t + RP_t) dt + s_t dz_t,$$

in which $s_t dz_t$ is the error term (the standard deviation $s_t$ is stochastic).

Rearrange, multiply by the discount factor, and integrate:

$$\int_t^\infty e^{-R(\tau-t)} (S_\tau d\tau - s_\tau dz_\tau) = \int_t^\infty e^{-R(\tau-t)} (RP_\tau d\tau - dP_\tau).$$

Then take the expectation at time $t$. 

Hence theorem (10) follows.