Two Equivalent Conditions

The traditional theory of present value puts forward two equivalent conditions for asset-market equilibrium:

Rate of Return The expected rate of return on an asset equals the market interest rate;

Present Value The asset price equals the present value of expected future payments.

We explain these two conditions and show that they are equivalent—either condition implies the other.

Market Interest Rate

The rate-of-return condition says just that all assets share a common expected rate of return. The market interest rate refers to the expected rate of return common to all assets.

We assume that the market interest rate $R > 0$ is constant.

Notation

Consider an asset with payment $\$_t$ at time t. For a stock, the payment would be the dividend. For a bond, the payment would be interest or principal.

Let P_t denote the asset price at time t.

Expected Rate of Return

Definition 1 (Return) *The return is the profit divided by the amount invested.*

Definition 2 (Expected Rate of Return) *The expected rate of return is the expected return divided by the length of the time period.*

Disequilibrium

If the expected rate of return were greater than the market interest rate, the security would be seen as a “good buy.” Investors would like to buy the security; those holding the security would not want to sell it. Demand would exceed supply. The reverse inequality would lead to excess supply.

Present Value

Definition 3 (Present Value) *The present value of a payment to be received in the future is the dollars attainable now by borrowing against the future payment.*

Definition 4 (Discount Factor) *The present value is the future payment multiplied by the discount factor.*

Discrete Time

With compound interest, a dollar borrowed at time 0 will require a repayment of $(1+R)^t$ at time t, the principal plus interest.

Theorem 5 (Present Value) *The present value at time* 0 *of* $\$_t$ *dollars at time* t *is*

$$\frac{\$_t}{(1+R)^t}$$

dollars. The discount factor is $1/(1+R)^t$.

The present-value equilibrium condition asserts that the asset price at time 0 equals the present value of expected payments,

$$P_0 = \$_0 + \frac{\mathrm{E}_0\left(\$_1\right)}{1+R} + \frac{\mathrm{E}_0\left(\$_2\right)}{(1+R)^2} + \frac{\mathrm{E}_0\left(\$_3\right)}{(1+R)^3} + \cdots .$$

Simple Example of Equivalence

Consider an asset paying P_1 at time 1 and paying nothing at other times. Suppose that the interest rate is R. What would be a fair price P_0 to pay for the asset at time 0?

Rate-of-Return Condition

Using the rate-of-return condition, what would be a fair price P_0 to pay for the asset at time 0? Setting the rate of return equal to the market interest rate gives

$$\frac{P_1 - P_0}{P_0} = R;$$

the profit is the capital gain. Solving for the price gives

$$P_0 = \frac{P_1}{1+R}. \tag{1}$$

Present-Value Condition

For this asset, the present-value condition says that the market price equals the present value of expected payments,

$$P_0 = \frac{P_1}{1+R}.$$

But this condition is identical to (1), obtained from the rate-of-return condition.

Continuous Time

Here $\$_t$ is the payment flow.

For an investment from time t to time $t + \mathrm{d}t$, the profit is the payment $\$_t\,\mathrm{d}t$ plus the capital gain $\mathrm{d}P_t$. The return during the period is the profit divided by the beginning-of-period price,

$$\frac{\$_t\,\mathrm{d}t + \mathrm{d}P_t}{P_t}.$$

Rate-of-Return Equilibrium Condition

Condition 6 (Rate-of-Return Equilibrium Condition) *The expected rate of return equals the market interest rate,*

$$\frac{\$_t\,\mathrm{d}t + \mathrm{E}_t\left(\mathrm{d}P_t\right)}{P_t} = R\,\mathrm{d}t. \tag{2}$$

Present Value

Consider an investment worth P_t at time t. If the investment earns the market interest rate R, then with continuous compounding its value follows the differential equation

$$\mathrm{d}P_t = RP_t\,\mathrm{d}t\,,$$

with the solution

$$P_t = P_0 \mathrm{e}^{Rt}\,.$$

One dollar invested at time 0 is worth e^{Rt} dollars at time t. Conversely, if one borrows e^{-Rt} dollars at time 0, with interest one will owe 1 dollar at time t.

Theorem 7 (Present Value) *The present value at time* 0 *of one dollar at time* t *is* e^{-Rt} *dollars, and the discount factor is* e^{-Rt}.

The present-value condition for asset-market equilibrium asserts that the asset price equals the present value of expected payments,

$$P_0 = \int_0^\infty \mathrm{e}^{-Rt} \mathrm{E}_0 (\$_t)\, \mathrm{d}t. \tag{3}$$

Equivalence

The rate-of-return condition (2) is equivalent to the present-value condition (3).

We first demonstrate the equivalence in several examples and then give a general proof.

Perpetual Bond

Consider a *perpetual bond*, a bond paying one-dollar interest per period in perpetuity; the principal is never repaid. If the interest rate is R, what is a fair price for the bond?

A fair price is

$$\frac{1}{R}, \tag{4}$$

for the bond then has rate of return R. For example, if $R = .10$, then the bond should sell for \$10, so it will return 10%. A lower price would give a higher yield, and a higher price would give a lower yield.

Present Value

To show the equivalence between the two equilibrium conditions, we must show that the present value of the payments is (4).

The present value is

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-Rt}\$_t\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-Rt}1\,\mathrm{d}t \\
&= -\frac{1}{R}\mathrm{e}^{-Rt}\Big]_0^\infty \\
&= \frac{1}{R}\mathrm{e}^{-R0} - \frac{1}{R}\lim_{t\to\infty}\mathrm{e}^{-Rt} \\
&= \frac{1}{R},
\end{aligned}$$

in accord with the expected rate-of-return reasoning.

Stock

Next we analyze a related but more complex example.

Consider a stock paying dividend D_t, and the dividend grows at the constant rate G. Here

$$\$_t = D_t = D_0 e^{Gt}.$$

What is a fair price P_0 for the stock at time 0?

Rate-of-Return for the Stock

The rate of return is the dividend yield plus the rate of capital gain. The dividend yield is D_t/P_t. Since the dividend grows each period at rate G, intuitively the stock price should also grow at this rate:

$$\frac{\mathrm{d}P_t}{P_t} = G\,\mathrm{d}t;$$

the rate of capital gain is G. The return is therefore

$$\frac{D_t\,\mathrm{d}t + P_t G\,\mathrm{d}t}{P_t}.$$

Setting the rate of return equal to the market interest rate gives

$$\frac{D_t}{P_t} + G = R.$$

Solving for P_t yields

$$P_t = \frac{D_t}{R - G}. \tag{5}$$

The formula makes sense qualitatively: raising D_t, reducing R, or raising G should increase P_t.

Present Value for the Stock

This price equals the present value at time 0 of the payments:

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-Rt}\$_t\,\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-Rt}\left(D_0\mathrm{e}^{Gt}\right)\mathrm{d}t \\
&= -\frac{D_0}{R-G}\mathrm{e}^{-(R-G)t}\Big]_0^\infty \\
&= \frac{D_0}{R-G}\mathrm{e}^{-(R-G)0} - \frac{D_0}{R-G}\lim_{t\to\infty}\mathrm{e}^{-(R-G)t} \\
&= \frac{D_0}{R-G},
\end{aligned}$$

in agreement with (5).

This result requires $R > G$. Without this condition, the dividend rises faster than the discount factor falls, and the present value is infinite.

General Equivalence

In general, the two conditions for equilibrium are equivalent. If the price equals the present value at every moment, then the rate of return equals the market interest rate at every moment; and *vice versa*. We prove the equivalence.

Condition 8 (Present-Value Equilibrium Condition) *The asset price equals the present value of expected payments,*

$$P_t = \int_t^{\infty} \mathrm{e}^{-R(\tau - t)} \mathrm{E}_t\left(\$_\tau\right) \mathrm{d}\tau. \tag{6}$$

Theorem 9 *If the present-value equilibrium condition (6) holds at every moment, then the rate-of-return equilibrium condition (2) holds at every moment.*

Theorem 10 *f the rate-of-return equilibrium condition (2) holds at every moment, then*

$$P_t = \int_t^\infty \mathrm{e}^{-R(\tau - t)} \mathrm{E}_t\left(\$_\tau\right) \mathrm{d}\tau + \lim_{\tau\to\infty} \left[\mathrm{e}^{-R(\tau-t)} \mathrm{E}_t\left(P_\tau\right)\right].$$

Condition 11 (No-Bubble) *As the future time goes to infinity, the present value of the expected future price goes to zero:*

$$\lim_{\tau\to\infty} \left[\mathrm{e}^{-R(\tau-t)} \mathrm{E}_t\left(P_\tau\right)\right] = 0. \tag{7}$$

Corollary 12 *If the rate-of-return equilibrium condition (2) holds at every moment, and the no bubble condition (7) holds, then the present-value equilibrium condition (6) holds at every moment.*

Bubble

A *bubble* refers to a situation in which the asset price is not set by expected future payments but instead is driven by the expectation of high capital gains. People pay a high price for the asset because its price is rising and they hope for further increases. The prospects for future payments are unimportant, as the asset owner hopes to sell the asset to someone else at a high price.

The terminology comes from the soap bubbles blown by children. The bubbles have nothing inside, and soon they pop. An asset bubble pops at some point, and the price falls.

In the analysis here, bubble refers to a situation in which the asset price exceeds the present value of expected payments. The no-bubble condition rules out this possibility, as the price and the present value are then equal.

No Uncertainty

For simplicity, first we assume no uncertainty.

First, assume (6): at every moment the price equals the present value.

We use the formula for the differentiation of an integral:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{a(t)}^{b(t)} f(\tau,t)\,\mathrm{d}\tau\right] = b'(t)\,f[b(t),t] - a'(t)\,f[a(t),t] + \int_{a(t)}^{b(t)} \frac{\partial f(\tau,t)}{\partial t}\,\mathrm{d}\tau.$$

In the integral, t appears three times, so the derivative has three terms.

Differentiating (6) obtains

$$
\begin{aligned}
\mathrm{d}P_t &= \mathrm{d}\left[\int_t^{\infty} \mathrm{e}^{-R(\tau-t)} \$_\tau \,\mathrm{d}\tau\right] \\
&= (-1)\,\mathrm{e}^{-R(t-t)} \$_t \,\mathrm{d}t + R\left[\int_t^{\infty} \mathrm{e}^{-R(\tau-t)} \$_\tau \,\mathrm{d}\tau\right] \mathrm{d}t \\
&= -\$_t \,\mathrm{d}t + RP_t \,\mathrm{d}t .
\end{aligned}
$$

Rearranging obtains the rate-of-return condition (2).

Reverse Implication

Conversely, suppose that the return equals the market return at every moment, and we work backwards by integrating.

Write (2) as

$$\$_t \, \mathrm{d}t = RP_t \, \mathrm{d}t - \mathrm{d}P_t \, .$$

Multiply by the discount factor and integrate,

$$
\begin{aligned}
\int_t^\infty \mathrm{e}^{-R(\tau-t)} \$_\tau \,\mathrm{d}\tau &= \int_t^\infty \mathrm{e}^{-R(\tau-t)} \left(RP_\tau \,\mathrm{d}\tau - \mathrm{d}P_\tau\right) \\
&= \int_t^\infty \mathrm{d}\left[-\mathrm{e}^{-R(\tau-t)} P_\tau\right] \\
&= -\mathrm{e}^{-R(\tau-t)} P_\tau \Big]_t^\infty \\
&= P_t - \lim_{\tau\to\infty} \mathrm{e}^{-R(\tau-t)} P_\tau .
\end{aligned}
$$

We have proved (10) for the case of no uncertainty.

Uncertainty

We extend the proof to uncertainty.

First, assume (6): at every moment the price equals the present value. Then

$$
\begin{aligned}
\mathrm{d}P_t &= \mathrm{d}\left[\int_t^\infty \mathrm{e}^{-R(\tau-t)} \mathrm{E}_t\left(\$_\tau\right) \mathrm{d}\tau\right] \\
&= \left[-\mathrm{e}^{-R(t-\tau)} \mathrm{E}_t\left(\$_\tau\right)\big|_{\tau=t}\right] \mathrm{d}t + \left[\int_t^\infty \mathrm{e}^{-R(\tau-t)} R\mathrm{E}_t\left(\$_\tau\right) \mathrm{d}\tau\right] \mathrm{d}t \\
&\quad + \int_t^\infty \left\{\mathrm{e}^{-R(\tau-t)} \mathrm{d}\left[\mathrm{E}_t\left(\$_\tau\right)\right]\right\} \mathrm{d}\tau \\
&= \left(-\$_t + RP_t\right) \mathrm{d}t + \int_t^\infty \mathrm{e}^{-R(\tau-t)} \left[\mathrm{E}_{t+\mathrm{d}t}\left(\$_\tau\right) - \mathrm{E}_t\left(\$_\tau\right)\right] \mathrm{d}\tau.
\end{aligned}
$$

Take the expected value:

$$
\begin{aligned}
\mathrm{E}_t\left(\mathrm{d}P_t\right) = {} & \left(-\$_t + RP_t\right)\mathrm{d}t \\
& + \int_t^{\infty} \mathrm{e}^{-R(\tau - t)} \mathrm{E}_t\left[\mathrm{E}_{t+\mathrm{d}t}\left(\$_\tau\right) - \mathrm{E}_t\left(\$_\tau\right)\right] \mathrm{d}\tau \\
= {} & \left(-\$_t + RP_t\right)\mathrm{d}t.
\end{aligned}
$$

Compared with the certainty case there is an extra term, but this extra term is zero:

$$
\mathrm{E}_t\left[\mathrm{E}_{t+\mathrm{d}t}\left(\$_\tau\right) - \mathrm{E}_t\left(\$_\tau\right)\right] = 0.
$$

The change in the expected value is an innovation, and the expected value of an innovation is zero. Rearranging obtains (2).

Alternative Derivation

$$
\begin{aligned}
P_t &= \int_t^{\infty} e^{-R(\tau-t)} E_t(\$_\tau)\, d\tau \\
&= \int_t^{t+dt} e^{-R(\tau-t)} E_t(\$_\tau)\, d\tau + \int_{t+dt}^{\infty} e^{-R(\tau-t)} E_t(\$_\tau)\, d\tau \\
&= e^{-R(t-t)} E_t(\$_t)\, dt + e^{-R\,dt} \int_{t+dt}^{\infty} e^{-R[\tau-(t+dt)]} E_t(\$_\tau)\, d\tau \\
&= \$_t\, dt + e^{-R\,dt} E_t \left\{ \int_{t+dt}^{\infty} e^{-R[\tau-(t+dt)]} E_{t+dt}(\$_\tau)\, d\tau \right\},
\end{aligned}
$$

since

$$
E_t\left[E_{t+dt}(\$_\tau)\right] = E_t(\$_\tau).
$$

Since the expression in braces is $P_{t+\mathrm{d}t}$,

$$
\begin{aligned}
P_t &= \$_t\,\mathrm{d}t + \mathrm{e}^{-R\,\mathrm{d}t}\mathrm{E}_t\left(P_{t+\mathrm{d}t}\right) \\
&= \$_t\,\mathrm{d}t + (1 - R\,\mathrm{d}t)\left[P_t + \mathrm{E}_t\left(\mathrm{d}P_t\right)\right] \\
&= \$_t\,\mathrm{d}t + (1 - R\,\mathrm{d}t)\,P_t + \mathrm{E}_t\left(\mathrm{d}P_t\right) \text{ as } \mathrm{d}t\,\mathrm{d}P_t = 0,
\end{aligned}
$$

which simplifies to (2).

Reverse Implication

By (2), then

$$\mathrm{d}P_t = \left(-\$_t + RP_t\right)\mathrm{d}t + s_t\,\mathrm{d}z_t\,,$$

in which $s_t\,\mathrm{d}z_t$ is the error term (the standard deviation s_t is stochastic).

Rearrange, multiply by the discount factor, and integrate:

$$\int_t^\infty \mathrm{e}^{-R(\tau-t)}\left(\$_\tau\,\mathrm{d}\tau - s_\tau\,\mathrm{d}z_\tau\right) = \int_t^\infty \mathrm{e}^{-R(\tau-t)}\left(RP_\tau\,\mathrm{d}\tau - \mathrm{d}P_\tau\right).$$

Then take the expectation at time t.

The expected value of the left-hand side is

$$\mathrm{E}_t\left[\int_t^\infty \mathrm{e}^{-R(\tau-t)}\left(\$_\tau\,\mathrm{d}\tau - s_\tau\,\mathrm{d}z_\tau\right)\right] = \int_t^\infty \mathrm{e}^{-R(\tau-t)}\mathrm{E}_t\left(\$_\tau\right)\mathrm{d}\tau,$$

as

$$\mathrm{E}_t\left(s_\tau\,\mathrm{d}z_\tau\right) = 0.$$

The right-hand side is

$$\begin{aligned}\int_t^{\infty} \mathrm{e}^{-R(\tau-t)}\left(RP_\tau\, \mathrm{d}\tau - \mathrm{d}P_\tau\right) &= \int_t^{\infty} \mathrm{d}\left[-\mathrm{e}^{-R(\tau-t)}P_\tau\right] \\ &= \left.-\mathrm{e}^{-R(\tau-t)}P_\tau\right]_t^{\infty} \\ &= P_t - \lim_{\tau\to\infty} \mathrm{e}^{-R(\tau-t)}P_\tau .\end{aligned}$$

Taking the expected value,

$$\mathrm{E}_t\left[\int_t^{\infty} \mathrm{e}^{-R(\tau-t)}\left(RP_\tau\, \mathrm{d}\tau - \mathrm{d}P_\tau\right)\right] = P_t - \lim_{\tau\to\infty} \mathrm{e}^{-R(\tau-t)}\mathrm{E}_t\left(P_\tau\right).$$

Hence theorem (10) follows.