Neoclassical One-Sector Growth Model

Consider the Solow neoclassical one-sector growth model with Cobb-Douglas production function

\[ Y = F(K, L) = K^{\frac{1}{3}} L^{\frac{2}{3}}. \]

Gross saving is \( sY \), with \( s = .12 \). The rate of population growth \( n = .03 \). Initially the capital/labor ratio \( k = K/L = 4 \).
Intensive Production Function

Because returns to scale are constant, output per capita is

\[ \frac{F(K, L)}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = F(k, 1) = f(k). \]

Applying this relationship to the production function here,

\[ f(k) = F(k, 1) = k^{\frac{1}{3}} \cdot 1^{\frac{2}{3}} = k^{\frac{1}{3}}. \]
Change in the Capital/Labor Ratio

Expressed *per capita*, capital deepening $\frac{dk}{dt}$ equals saving $sf(k)$ less capital widening $nk$:

$$\frac{dk}{dt} = sf(k) - nk.$$
The capital widening $dk/dt$ is the increase in capital per capita.

Since national income equals national product, income per capita equals output per capita $f(k)$. Saving per capita $sf(k)$ is income per capita times the fraction of income saved.

Part of the saving is used to equip new workers with capital. The population growth rate $n$ is the number of new workers per capita. Each worker requires $k$ units of capital, so saving per capita for capital widening is $nk$.

The remainder of the saving is available for capital deepening, increasing the capital per capita. This residual saving per capita is $sf(k) - nk$, so capital per capita goes up by this amount.
For the model here,

\[
\frac{dk}{dt} = sf(k) - nk = .12k^{\frac{1}{3}} - .03k,
\]

and the Solow diagram shows the relationship.
Macroeconomics

Solow Growth Model—Example

\[ 0.03k \]

\[ 0.12k^{1/3} \]
Short-Run Behavior

For the initial value $k = 4$, the figure shows that saving exceeds capital widening, so capital deepening occurs. The capital/labor ratio rises:

$$\frac{dk}{dt} = .12k^{\frac{1}{3}} - .03k$$

$$= .12(4)^{\frac{1}{3}} - .03 \times 4$$

$$\approx .07.$$
Long-Run Behavior

In the long run, the economy converges to steady-state growth. The capital/labor ratio is constant:

\[ 0 = \frac{dk}{dt} = sf(k) - nk = .12k^{\frac{1}{3}} - .03k, \]

with solution \( k = 8 \).

*Per capita* values are constant. The growth rate of output, capital, consumption, and investment are all constant at the rate of population growth, \( n = .03 \).
Higher Saving Rate

Alternatively, suppose that the saving fraction is $s = .27$. In the Solow diagram, saving per capita rises.
Macroeconomics

Solow Growth Model—Example

![Graph showing Capital/Labor vs Output/Labor with various lines representing different growth models.](image)
Short-Run Behavior

In the short run, saving is higher. The capital/labor ratio increases more rapidly, and higher saving and investment cause faster output growth.

For the initial value \( k = 4 \),

\[
\frac{dk}{dt} = .27k^{\frac{1}{3}} - .03k
\]

\[
= .27(4)^{\frac{1}{3}} - .03 \times 4
\]

\[
\approx .31.
\]
Long-Run Behavior

In the long run, the economy again converges to steady-state growth, but the capital/labor ratio is higher. In steady-state growth,

\[ 0 = \frac{dk}{dt} = sf(k) - nk = .27k^{\frac{1}{3}} - .03k, \]

with solution \( k = 27 \).

*Per capita* values are constant, but output *per capita* is higher with higher saving. Again the population growth \( n = .03 \) determines the growth rate of output, capital, consumption, and investment.