CHARACTERS, SUPERCHARACTERS AND WEBER MODULAR FUNCTIONS

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Abstract. We study certain canonical automorphic forms associated to irreducible characters of \( N = 1 \) superconformal minimal models in both the Neveu-Schwarz and Ramond sector by using representation theoretic methods. By extending the techniques from [35] to the setting of \( N = 1 \) superconformal algebras we obtain a series of modular identities involving certain Wronskian determinants in both Neveu-Schwarz (untwisted) and Ramond (twisted) sector. Three Weber modular functions play a fundamental role here as well as the Dedekind \( \eta \)–function. In the most interesting case of \( SM(2, 4k) \) minimal series, we obtain a derivation and generalizations of several classical modular \( q \)–series identities (e.g., Jacobi’s Four Square Theorem, a Carlitz’ identity, specialized Macdonald’s identities for \( B \) series, etc.).

0. Introduction and notation

The notion of a vertex operator superalgebra is a mild, but important generalization of the notion of vertex operator algebra. If a vertex operator superalgebra is equipped with an action of the \( N = 1 \) Neveu-Schwarz Lie superalgebra, the corresponding structure is called \( N = 1 \) vertex operator superalgebra [13], [22], [27]. As in the case of vertex operator algebras, the most interesting class of vertex operator superalgebras are the rational ones [42] (cf. [22]). Rational vertex operator algebras share several important features such as the modular invariance of graded dimensions (or characters) [42]. On the other hand modular invariance for rational vertex operator superalgebras fails if we work solely with characters of irreducible modules. In order to assure the modular invariance one takes into account characters of irreducible \( \sigma \)–twisted modules (cf. [11]), where \( \sigma \) is the parity automorphism, and supergraded dimensions (or supercharacters) of ordinary modules.

Among the most important models of \( N = 1 \) superconformal field theories are those associated to \( N = 1 \) minimal models [3], [10], [20], [21], [22]. These models have been studied from the vertex operator superalgebra point of view in [1], [2], [19], [22], [27], [28], etc.

In the present work we focus on number theoretic properties of irreducible characters of \( N = 1 \) superconformal minimal models. Our present work forms a natural extension of [35], so we refer the reader to the introduction of [35] for our motivation (see also [34]).

Let us introduce some notation first. Let \( q = e^{2\pi i \tau} \), where \( \tau \in \mathbb{H} \), being the upper half-plane. The Dedekind \( \eta \)–function is usually defined as the infinite product

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]

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an automorphic form of weight $\frac{1}{2}$. We will also make use of three Weber modular functions $^1$:

\begin{align*}
\text{(0.1)} & \quad f(\tau) = q^{-1/48} \prod_{n=0}^{\infty} (1 + q^{n+1/2}) , \\
\text{(0.2)} & \quad f_1(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 + q^n) , \\
\text{(0.3)} & \quad f_2(\tau) = q^{-1/48} \prod_{n=1}^{\infty} (1 - q^{n-1/2}) .
\end{align*}

It is known that the vector space spanned by Weber functions is modular invariant (i.e., an $SL(2, \mathbb{Z})$-module) and that appropriately normalized $24$th powers of $f$, $f_1$ and $f_2$ are the roots of $(X - 16)^3 - j(\tau)X = 0$, where $j(\tau) = q^{-1} + 744 + \cdots$ is the Klein modular invariant (for more about Weber functions and applications in computing class invariants see [9], [41]).

In conformal field theory the Weber modular functions naturally arise in the setup of the free fermion vertex operator superalgebra of the central charge $\frac{1}{2}$ (i.e., the Ising model [13]). Throughout, we will be using Ramanujan’s derivative

\[ \tau' = \frac{1}{2\pi i} d\tau = \left( \frac{d}{dq} \right). \]

The logarithmic derivative of $\eta(\tau)$ is then the second Eisenstein series

\[ E_2(\tau) = \frac{1}{24} + \sum_{n=1}^{\infty} \frac{nj^n}{1 - q^n} , \]

which is a quasimodular form for $SL(2, \mathbb{Z})$. Moreover, logarithmic derivatives of $f(\tau)$, $f_1(\tau)$ and $f_2(\tau)$ are given by

\begin{align*}
E_{2,0}(\tau) &= \frac{-1}{48} + \sum_{n=1}^{\infty} \frac{(n - 1/2)q^{n-1/2}}{1 + q^{n-1/2}} , \\
E_{2,1}(\tau) &= \frac{1}{24} + \sum_{n=1}^{\infty} \frac{nj^n}{1 + q^n} , \\
E_{2,2}(\tau) &= \frac{-1}{48} - \sum_{n=1}^{\infty} \frac{(n - 1/2)q^{n-1/2}}{1 - q^{n-1/2}} ,
\end{align*}

(0.4) respectively. The series $E_{2,1}(\tau)$ is an ordinary modular form of weight 2 for $\Gamma_0(2)$ [7], and $E_{2,0}(\tau)$ and $E_{2,2}(\tau)$ are modular forms of weight two for $\Gamma(48) \subset \Gamma(1)$.

Let us elaborate our main results. To every rational $N = 1$ vertex operator superalgebra associated to $N = 1$ minimal models at level $c_{p,p'}$, we associate three Wronskian determinants; the first one is associated to irreducible characters in the Neveu-Schwarz (NS) sector, the second is associated to irreducible characters in the Ramond (R) sector and the third to supercharacters in the NS sector. Interestingly, these automorphic forms can be studied by using representation theoretic methods. Thus, we obtain an $N = 1$ version of all our results from [34] and [35].

We have $^1f_1$ is usually multiplied by $\sqrt{2}$, or even 2. From our point of view this normalization is irrelevant.
Theorem 0.1. For every \( p, p' \in \mathbb{N} \) such that \( \gcd\left(\frac{p-p'}{2}, p\right) = 1 \), let
\[
|c_{p,p'}| = \frac{(p-1)(p'-1)}{4} + \frac{1-\nu}{4} \in \mathbb{N}.
\]
Also, let \( \chi_1(\tau), \ldots, \chi_{|c_{p,p'}|}(\tau) \) be a list of all irreducible characters associated to \( N = 1 \) \((p, p')\)-minimal models in the NS sector. Then
\[
W_{\left(\frac{q}{\mathbb{Z}}\right)}(\chi_1(\tau), \ldots, \chi_{|c_{p,p'}|}(\tau)) = \eta(\tau)^{2|c_{p,p'}|(|c_{p,p'}|-1)}f(\tau)^3|c_{p,p'}|,
\]
for \( \nu = 1 \) and
\[
W_{\left(\frac{q}{\mathbb{Z}}\right)}(\chi_1(\tau), \ldots, \chi_{|c_{p,p'}|}(\tau)) = \frac{\eta(\tau)^{2|c_{p,p'}|(|c_{p,p'}|-1)}}{f(\tau)^{|c_{p,p'}|-1}},
\]
for \( \nu = 0 \). Here \( W_{\left(\frac{q}{\mathbb{Z}}\right)} \) is an appropriately normalized Wronskian determinant.

Let \( (\frac{\cdot}{q}) \) denote the Kronecker symbol (i.e., \( (\frac{n}{8}) = 1 \), for \( n \equiv \pm 1 \mod 8 \), \( (\frac{n}{8}) = -1 \), for \( n \equiv \pm 3 \mod 8 \), and zero otherwise). The following formula was proven by Carlitz \cite{8} in 1953. In our setup it is simply a consequence of Theorem 0.1, applied in the \((p, p') = (2, 8)\) case.

Corollary 0.2. (Carlitz, "N=1 Ramanujan" \textsuperscript{2})
\[
(0.5) \quad 1 - 2 \sum_{n=1}^{\infty} \left(\frac{n}{2}\right) \frac{nq^n}{1-q^n} = \frac{\eta^3(4z)\eta(2\tau)\eta^2(\tau)}{\eta^2(8\tau)}.
\]

In the Ramond sector we have an analogue of Theorem 0.1.

Theorem 0.3. For every \( p, p' \in \mathbb{N} \) such that \( \gcd\left(\frac{p-p'}{2}, p\right) = 1 \) and \( p, p' \in 2\mathbb{Z} + \nu \), where \( \nu \in \{0, 1\} \), let
\[
|c_{p,p'}| = \frac{(p-1)(p'-1)}{4} + \frac{1-\nu}{4} \in \mathbb{N}.
\]
Also, let \( \chi_1(\tau), \ldots, \chi_{|c_{p,p'}|}(\tau) \) be a list of irreducible characters associated to \( N = 1 \) \((p, p')\)-minimal models in the R sector. Then
\[
W_{\left(\frac{q}{\mathbb{Z}}\right)}(\chi_1(\tau), \ldots, \chi_{|c_{p,p'}|}(\tau)) = \eta(\tau)^{2|c_{p,p'}|(|c_{p,p'}|-1)}f_1(\tau)^3|c_{p,p'}|,
\]
for \( \nu = 1 \) and
\[
W_{\left(\frac{q}{\mathbb{Z}}\right)}(\chi_1(\tau), \ldots, \chi_{|c_{p,p'}|}) = \frac{\eta(\tau)^{2|c_{p,p'}|(|c_{p,p'}|-1)}}{f_1(\tau)^{|c_{p,p'}|-1}},
\]
for \( \nu = 0 \).

A consequence of Theorem 0.3, in the \((p, p') = (2, 8)\) case is the famous Jacobi’s Four Square Theorem.

\textsuperscript{2}The formula in Corollary 0.2 is an \( N = 1 \) supersymmetric analogue of a Ramanujan’s modulus 5 identity studied in \cite{35}, whereas in the Ramanujan’s formula the Kronecker symbol is replaced by the Dirichlet character \( (\frac{\cdot}{q}) \) (Legendre symbol).
Corollary 0.4.

\[(0.6) \quad \left( \sum_{n \in \mathbb{Z}} (-q)^{n^2} \right)^4 = 1 + 8 \left( \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1+q^{2n}} - \frac{(2n-1)q^{2n-1}}{1+q^{2n-1}} \right). \]

Three Weber functions satisfy the following two fundamental relations:

\[(0.7) \quad f(\tau)f_1(\tau)f_2(\tau) = 1 \]

\[(0.8) \quad W_{q,\frac{1}{\pi}}(f(\tau), f_1(\tau), f_2(\tau)) = \eta(\tau)^{12} \]

where the second formula was proven in [34].

If we combine Theorems 0.1 and 0.3, and an analogous result for supercharacters (see Theorem 6.1) we have the following analogue of (0.7)-(0.8):

**Proposition 0.5.** For \( p, p' \in \mathbb{N}, \ g.c.d. \left( \frac{p-p'}{2}, p \right) = 1, \) we have

\[(0.9) \quad W_{p,p'}^{NS}(\tau) \cdot W_{p,p'}^{R}(\tau) \cdot W_{p,p'}^{F}(\tau) = \eta(\tau)^{6k^2-6k}, \]

\[(0.10) \quad W_{q,\frac{1}{\pi}}(W_{p,p'}^{NS}(\tau), W_{p,p'}^{R}(\tau), W_{p,p'}^{F}(\tau)) = \eta(\tau)^{6k^2-6k+12}, \]

where \( k = |c_{p,p'}|, \) and \( W_{p,p'}^{NS}(\tau) \) stands for the Wronskian in Theorem 0.1, \( W_{p,p'}^{R}(\tau) \) for the Wronskian in Theorem 0.3 and \( W_{p,p'}^{F}(\tau) \) for the Wronskian associated to supercharacters in the NS sector.

In Appendix A we obtained certain recursion formulas needed for the proof of Theorem 5.1. The proofs there are rather technical and rely on ideas from [11], [33] and [42] so we felt that they belong outside the main text. These formulas can be also used for different superconformal models. Finally, in Appendix B we prove \( \sigma \)-rationality for \( N = 1 \) minimal models needed in Theorem 5.1 and ultimately for proving Theorem 0.3.

**n.b.** This work was presented at the conference *Additive Number Theory*, Gainesville, November 2004. We would like to thank the organizer for the invitation. Also, recently we noticed a related paper [12], where the authors extended results from [11] to the setup of vertex operator superalgebras.

1. \( N = 1 \) Vertex Operator Superalgebras

In this section we gather a few basic results about \( N = 1 \) vertex operator superalgebras. For a good introduction to this structure and important results see [27], [28], [22] and [40]. The geometric counterpart of \( N = 1 \) vertex operator algebra theory was carried out in [4]. The whole subject is deeply related to supergeometry on supercurves [30].

Let \( \text{Vir}_0 \) and \( \text{Vir}_{1/2} \) denote the \( N = 1 \) Ramond and \( N = 1 \) Neveu-Schwarz algebra, respectively. The brackets for \( \text{Vir}_\epsilon, \epsilon \in \{0, \frac{1}{2}\} \) are given by

\[[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}C, \]
\[[G_m, L_n] = (m - \frac{n}{2})G_{m+n}, \]
\[[G_r, G_s] = 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}C, \]
where $C$ is the central element and $m \in \mathbb{Z}$, $n, r, s \in \mathbb{Z} + \epsilon$.

For $\epsilon = \frac{1}{2}$ the algebra is $\frac{1}{2}\mathbb{Z}$-graded in the obvious way, while for $\epsilon = 0$ the algebra is $\mathbb{Z}$-graded (in both cases the central element $C$ has degree zero). We will use Vir$_{-,\epsilon}$, Vir$_{+,\epsilon}$ and Vir$_{0,\epsilon}$ to denote subalgebras of Vir$_{\epsilon}$ spanned by negative, positive and zero degree generators, respectively. If Vir$_{\epsilon}$ is acting on a module, the operators corresponding to the generators will be denoted by $L(m)$, $G(n)$. The central element $C$ shall always act as the multiplication with a complex number $c$. If $M$ is a Vir$_{\epsilon}$-module we shall always write $M_{\epsilon}$ to avoid any confusion.

Let us recall some portions of the definition of $N = 1$ vertex operator superalgebra and $N = 1$ vertex operator superalgebra module as in [27] or [22] (the reader may also find [13], [4] or even [19] useful). All our vector spaces are now $\mathbb{Z}_2$-graded (super vector spaces), where for a super vector space $V$ the parity decomposition will be denoted by $V = V_0 \oplus V_1$, where $V_0$ is the even subspace and $V_1$ the odd subspace (horizontal grading). Informally, a vertex operator superalgebra is a quadruple $(V, Y, \omega, \tau, 1)$ where $V$ is a super vector space, equipped with a $\frac{1}{2}\mathbb{Z}$-grading (vertical grading),

$$V = \prod_{n \in \mathbb{Z}_2} V_n,$$

and the vertex operator map

$$Y : V \rightarrow \text{End}(V)[[x, x^{-1}]],$$

subject to certain grading conditions, the vacuum condition, the creation condition, the Virasoro axioms and the Jacobi identity. For simplicity we will also require that

$$V^0 = \prod_{n \in \mathbb{Z}} V_n, \quad V^1 = \prod_{n \in \mathbb{Z} + \frac{1}{2}} V_n.$$

In addition, the Virasoro algebra acts on $V$ and the operator $L(0)$ is compatible with the (vertical) grading of $V$.

An $N = 1$ vertex operator superalgebra $(V, Y, \omega, \tau, 1)$ is a vertex operator superalgebra with a distinguished (odd) vector

$$\tau \in V_{\frac{3}{2}}, \quad Y(\tau, x) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} G(n)x^{-n-3/2}$$

which, together with $\omega$, defines a representation of Vir$_{1/2}$. An $N = 1$ module is a $\mathbb{Z}_2$-graded vector space $M = M^0 \oplus M^1$, which is also $\mathbb{C}$-graded, equipped with the vertex operator map which is compatible with the $\mathbb{Z}_2$-grading of $V$, so that for every $u \in V^i$,

$$Y(u, x)M^j \in M^{i+j}[[x, x^{-1}]], \quad i, j \in \mathbb{Z}_2.$$

In addition, $M$ carries a representation of the $N = 1$ NS algebra of the same central charge as $V$. Besides the $Y$-vertex operators, as in [14], we will use slightly modified vertex operators

$$X(\cdot, x) = Y(x^{L(0)}, x).$$
If we attempt to write a closed formula for the (anti)commutator of two $X$–operators, we would discover the Zhu’s operators \[25\], \[33\], \[42\]
\[
Y[\cdot, x] := Y(e^{xL(0)}, e^{x} - 1), \quad Y[u, x] = \sum_{n \in \mathbb{Z}} u[n] x^{-n-1}.
\]

The following result is from \[33\] (see also \[18\], \[42\]):

**Theorem 1.1.** Let $(V, Y(\cdot, y), 1, \omega, \tau)$ be a $N = 1$ vertex operator superalgebra. Then $(V, Y(\cdot, y), 1, \tilde{\omega}, \tau)$, where $\tilde{\omega} = \omega - \frac{c}{24}$, has an $N = 1$ vertex operator superalgebra structure which is isomorphic to $(V, Y(\cdot, y), 1, \omega, \tau)$.

We should say here that the previous result can be substantially generalized if we use a geometric interpretation of $N = 1$ vertex operator superalgebras \[4\] \[5\] (see also \[18\]).

Besides ordinary modules a highly non-classical notion is that of a twisted module for vertex operator algebras (cf. \[15\], \[28\]). To define a twisted module we need a finite order automorphism $\nu$ of $V$, giving a decomposition

\[
V = \bigoplus_{i=0}^{T-1} V(i),
\]

where $V(i)$ is the eigenspace for $\nu$, with the eigenvalue $e^{2ki\pi i/T}$. Informally, a $\nu$–twisted module $M^\nu$ is then a $\mathbb{C}$-graded space equipped with a vertex operator map

\[
Y^\nu(\cdot, x) : V \rightarrow M^\nu[[x^{1/T}, x^{-1/T}]]
\]
satisfying several natural axioms including the twisted Jacobi identity (for details see \[28\]). In particular, for every $u \in V^{(r)}$

\[
Y^\nu(u, x) = \sum_{n \in \mathbb{Z}+r/T} u[n] x^{-n-1}.
\]

The notion of a twisted module notion can be easily generalized for vertex operator superalgebras \[28\], \[40\]. Unlike vertex operator algebras, every vertex operator superalgebra has a canonical automorphism of order two, the parity map $\sigma$ \[13\]. Explicitly,

\[
\sigma : V \rightarrow V,
\]

\[
\sigma(v) = v, \quad v \in V^0, \quad \sigma(v) = -v, \quad v \in V^1.
\]

Now, we can define $\sigma$–twisted $V$–modules as in \[13\], \[17\]. For several reasons (e.g., applications in string theory) $\sigma$–twisted operators are more fundamental and more interesting than ordinary modules. Let us examine the effect of twisting with $\sigma$ on the generators of the $N = 1$ superconformal algebra. From the Jacobi identity for twisted modules (cf. \[28\]) it follows that the Fourier coefficients of

\[
Y^\sigma(\tau, x) = \sum_{m \in \mathbb{Z}} G(m)x^{-m-3/2} \quad \text{and} \quad Y^\sigma(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}
\]
define a representation of the $N = 1$ Ramond superalgebra. Thus, every $\sigma$-twisted module for an $N = 1$ vertex operator algebra is also a module for the $N = 1$ Ramond Lie superalgebra (i.e., $Vir_0$). Because of that we will assume that every $\sigma$-twisted module for $N = 1$ vertex operator superalgebra is also $\mathbb{Z}_2$-graded.

In the setup of $N = 1$ vertex operator superalgebras we often refer to $\sigma$-twisted $V$–modules as modules in the Ramond sector. In parallel, it is customary to refer to untwisted $V$–modules as modules in the Neveu-Schwarz sector. This notation was introduced by physicists. An
irreducible module for an $N = 1$ superalgebra is called irreducible (or simple) if it does not contain a proper $\mathbb{Z}_2$-graded submodule.

Let $V$ be an $N = 1$ vertex operator superalgebra. Suppose further that $V$ has finitely many nonisomorphic modules $M_i$, $i \in I$, with irreducible characters

$$\text{tr}|_M q^{L(0)} - c/24,$$

being holomorphic functions in the upper half plane. Let us denote the vector space spanned by irreducible characters by $\mathcal{M}_V$ and its dimension by $d_V$. If a vertex operator superalgebra $V$ is fixed we will drop the subscript and write $d$ instead. Let $f_1, f_2, \ldots, f_d$ be a basis of $\mathcal{M}_V$. We define the Wronskian of $V$

$$W^{NS}_V(\tau)$$

as a multiple of

$$W_{q^{\frac{d}{2}}}(f_1, \ldots, f_d) := \left| \begin{array}{cccc}
  f_1 & f_2 & \cdots & f_d \\
  f'_1 & f'_2 & \cdots & f'_d \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(d-1)}_1 & f^{(d-1)}_2 & \cdots & f^{(d-1)}_d \\
\end{array} \right|,$$

uniquely determined by the condition that the first non-zero coefficient in the $q$-expansion is one. Suppose now that there are finitely many inequivalent irreducible $\sigma$–twisted modules whose characters are holomorphic in $\mathbb{H}$. We denote by $\mathcal{M}_{V}^\sigma$ the vector space spanned by irreducible characters. Let $g_1, \ldots, g_r$ be a basis of $\mathcal{M}_{V}^\sigma$. We define the Wronskian of $V$ in the Ramond sector

$$W^{R}_V(\tau)$$

as a multiple of

$$W_{q^{\frac{d}{2}}}(g_1(\tau), \ldots, g_r(\tau)),$$

with the same normalization property as for $W^{NS}_V(\tau)$ (in general, $r \neq d$). Let

$$\text{str}|_M q^{L(0)} = \text{tr}|_M q^{L(0)} - c/24,$$

denote the supercharacter of an untwisted $V$–module $M$. Finally, we will denote by

$$W^{F}_V(\tau)$$

the Wronskian associated to irreducible supercharacters in the NS sector.

In parallel with the vertex operator algebras we will now consider the “smallest” $N = 1$ vertex operator superalgebras, i.e., those associated to irreducible lowest weight modules for the $N = 1$ Neveu-Schwarz algebra [22].

### 2. $N = 1$ Superconformal Minimal Models

Let $p < p' \in \mathbb{N}$ such that

$$(2.13) \quad p' - p \in 2\mathbb{Z} \quad \text{and} \quad \gcd \left( \frac{p' - p}{2}, p \right) = 1.$$

In the NS sector (NS-sector) let

$$c_{p,p'} = \frac{3}{2} \left( 1 - \frac{2(p - p')^2}{pp'} \right), \quad h_{p,p'}^{r,s} = \frac{(sp - pr)^2 - (p - p')^2}{8pp'},$$
where $1 \leq r \leq p - 1$, $1 \leq s \leq p' - 1$, satisfying $r - s \in 2\mathbb{Z}$. In the Ramond sector (R-sector) $c_{p,p'}$ will be as above and

$$h^{r,s}_{p,p'} = \frac{(sp - p'r)^2 - (p - p')^2}{8pp'} + \frac{1}{16},$$

where $1 \leq r \leq p - 1$, $1 \leq s \leq p' - 1$, satisfying $r - s \in 2\mathbb{Z} + 1$. Set

$$(2.14) \quad \mathcal{H}_p \subset \{ h^{r,s}_{p,p'} : 1 \leq r \leq p - 1, 1 \leq s \leq p' - 1, r - s \in r - s \in 2\mathbb{Z} + (1 - 2\epsilon) \}.$$

It is not hard to see that the cardinality of $\mathcal{H}_p$ is

$$|c_{p,p'}| := \frac{(p - 1)(p' - 1) \pm 1 + (-1)^{pp'} \pm 1}{8},$$

for both $\epsilon = 0$ and $\epsilon = \frac{1}{2}$.

Let us denote by $L_\epsilon(c_{p,p'}, h^{r,s}_{p,p'})$ the irreducible (or simple) lowest weight module for $\text{Vir}_\epsilon$ with the lowest weight $h^{r,s}_{p,p'} \in \mathbb{Q}$ and central charge $c_{p,p'}$ [22], [27], [20], [21]. Keep in mind that $L_\epsilon(c_{p,p'}, h^{r,s}_{p,p'})$ are $\mathbb{Z}_2$-graded with respect to the action of $\text{Vir}_\epsilon$. Thus, for $h^{r,s}_{p,p'} \neq \frac{c_{p,p'}}{24}$, the dimension of the lowest graded subspace of $L_\epsilon(c_{p,p'}, h^{r,s}_{p,p'})$ is two and if $h^{r,s}_{p,p'} = \frac{c_{p,p'}}{24}$, then this dimension is one. For more details see [20],[21]. There are exactly $|c_{p,p'}|$ minimal models with the central charge $c_{p,p'}$. The following result is well-known (see for instance [6]):

**Proposition 2.1.** For $\epsilon \in \{0, \frac{1}{2}\}$

$$(2.15) \quad \text{tr}|_{L_\epsilon(c_{p,p'}, h^{r,s}_{p,p'})} q^{L(0)-\epsilon} = \frac{2(1-\epsilon)}{1+\delta_0,0} \frac{q^{h^{r,s}_{p,p'} - c_{p,p'}}}{\delta_{h,c/24}} \frac{(-q^{1-\epsilon})}{(q^\infty)} \sum_{j \in \mathbb{Z}} \left( q^{\frac{c_{p,p'} (pp' + pp' - s)}{2}} - q^{\frac{c_{p,p'} (pp' + pp' + s)}{2}} \right),$$

where $(a)_{\infty} = \prod_{i=0}^{\infty} (1 - aq^i)$ and $\delta$ is the Kronecker symbol.

We will make use of the following observation: For $p, p' \in 2\mathbb{N} + 1$,

$$\sum_{h \in \mathcal{H}_{p,p'}} \left( h - \frac{c_{p,p'}}{24} \right) = \frac{(pp' - p - p' + 1)(pp' - p - p' - 6)}{192},$$

and for $p, p' \in 2\mathbb{N}$,

$$\sum_{h \in \mathcal{H}_{p,p'}} \left( h - \frac{c_{p,p'}}{24} \right) = \frac{(pp' - p - p' + 3)(pp' - p - p' - 2)}{192}.$$

The purpose of the next two formulas will become clear later

$$(2.16) \quad \frac{(pp' - p - p' + 1)(pp' - p - p' - 6)}{192} = \frac{1}{24} \left( 2|c_{p,p'}|(|c_{p,p'}| - 1) + \frac{-1}{48} \left( 3|c_{p,p'}| \right) \right),$$

$$(2.17) \quad \frac{(pp' - p - p' + 3)(pp' - p - p' - 2)}{192} = \frac{1}{24} \left( 2|c_{p,p'}|(|c_{p,p'}| - 1) + \frac{-1}{48} \left( -|c_{p,p'}| + 1 \right) \right).$$
In the Ramond sector (i.e., \( \epsilon = 0 \)) we have a slightly different result. For \( p, p' \in 2\mathbb{N} + 1 \),
\[
\sum_{h \in \mathcal{H}_{p,p'}^0} (h - \frac{c_{p,p'}}{24}) = \frac{(pp' - p - p' + 1)(pp' - p - p' - 3)}{192},
\]
and for \( p, p' \in 2\mathbb{N} \),
\[
\sum_{h \in \mathcal{H}_{p,p'}^0} (h - \frac{c_{p,p'}}{24}) = \frac{(pp' - p - p')(pp' - p - p' - 2)}{192}.
\]
Again we will exploit the following identities
\[
\frac{(pp' - p - p' + 1)(pp' - p - p' - 3)}{192} = \frac{1}{24} \left( 2|c_{p,p'}|(|c_{p,p'} - 1|) + \frac{1}{24} \left( 3|c_{p,p'}| \right) \right)
\]
\[
\frac{(pp' - p - p')(pp' - p - p' - 2)}{192} = \frac{1}{24} \left( 2|c_{p,p'}|(|c_{p,p'} - 1|) + \frac{1}{24} \left( -|c_{p,p'}| + 1 \right) \right).
\]
As we mentioned earlier, for every untwisted irreducible \( L_{1/2}(c_{p,p'},0) \)-module \( M \) its supercharacter is defined as
\[
\text{str}|_M q^{L(0)-c/24} = \text{tr}|_M \sigma q^{L(0)-c/24}.
\]
From the character formulas (that are consequences of BGG-type resolutions in terms of Verma module \([3],[10],[21]\)) and the formula
\[
\sigma(v) = \begin{cases} 
  v, & \text{for } v \in L_{1/2}(c_{p,p'}, h)_{m+h}, m \in \mathbb{N} \\
  -v, & \text{for } v \in L_{1/2}(c_{p,p'}, h)_{m+h}, m \in \mathbb{N} + \frac{1}{2} 
\end{cases}
\]
it is easy to give explicit formulas for supercharacters starting from ordinary characters without any (additional) reference to representation theory of \( \text{Vir}_{1/2} \). Thus we can easily prove the following proposition as a consequence of BGG-type resolutions of \( L_{1/2}(c_{p,p'}, h) \) (see for instance \([20]\)).

**Proposition 2.2.**

For \( p, p' \in 2\mathbb{Z} \), we have
\[
\text{str}|_{L_{1/2}(c_{p,p'}, h,r,s)} q^{L(0)-c/24}
\]
\[
= q^{(h_{p,p'} - c_{p,p'}/24)(q^{1/2} - (q)_\infty \sum_{j \in \mathbb{Z}} \left( \frac{q^{j(pp'+s)}}{2} - (-1)^{rs} q^{(jp+r)(jp'+s)/2} \right)^2),}
\]
(2.18)

Let \( p, p' \in 2\mathbb{Z} + 1 \). Then
\[
\text{str}|_{L_{1/2}(c_{p,p'}, h,r,s)} q^{L(0)-c/24}
\]
\[
= q^{(h_{p,p'} - c_{p,p'}/24)(q^{1/2} - (q)_\infty \sum_{j \in \mathbb{Z}} \left( q^{2jp+2pp'+j(rp-sp')} - q^{2(j+1/2)^2pp'+(j+1/2)(rp-sp')} \right)^2)}
\]
\[
- (-1)^{rs} \sum_{j \in \mathbb{Z}} \left( q^{2jp+2pp'+j(rp+sp')+rs/2} - q^{2(j-1/2)^2pp'+(j-1/2)(rp-sp')} \right).}
\]
(2.19)
3. Vertex operator superalgebras associated to $N = 1$ superconformal minimal models

Let $V_{1/2}(c,0)$ be the vacuum module for $Vir_c$ (see [22], where it was denoted by $M_c$ or [27], where it was denoted by $\hat{M}(c,0)$). The corresponding irreducible quotient will be denoted by $L_{1/2}(c,p',0)$. Now, let us recall a result from [27] (see also [22]):

**Theorem 3.1.** The vacuum module $V_{1/2}(c,0)$ carries a canonical $N = 1$ vertex operator superalgebra with $\tau = G(-3/2)1$ (and $\omega = L(-2)1$).

In [27] it was also proven that every restricted module for the $N = 1$ NS algebra is a weak $V_{1/2}(c,0)$–module. In particular, for every $h \in \mathbb{C}$, $L_{1/2}(c,h)$ is an irreducible $V_{1/2}(c,0)$–module. Irreducible $L_{1/2}(c,p',0)$–modules were classified in [1], and partially in [22]. Fusion rings for $N = 1$ superconformal $(p,p')$–minimal models can be computed in two different ways: via coinvariants or by using Zhu’s theory (cf. [40], [42]). We have shown in [32] that the coinvariant approach is in fact equivalent to vertex operator superalgebra approach, at least in the case of $N = 1$ superconformal vertex operator superalgebra $L_{1/2}(c,0)$. In particular, a classification of irreducible $L_{1/2}(c,p',0)$–modules can be deduced from computations of certain coinvariants (see Section 9). We will need the following result:

**Theorem 3.2.**

(a) The vertex operator superalgebra $L_{1/2}(c,p',0)$ has finitely many equivalence classes of irreducible modules, with representatives $L_{1/2}(c,p', h_{p,p'}^{r,s})$, where $r = 1, \ldots, p - 1$ and $s = 1, \ldots, p' - 1$ satisfy $r - s = 2\mathbb{Z}$.

(b) Irreducible $\sigma$–twisted $L_{1/2}(c,p',0)$–modules are (up to equivalence) precisely irreducible modules in the Ramond sector $L_0(c,p', h_{p,p'}^{r,s})$, where $r = 1, \ldots, p$ and $s = 1, \ldots, p'$ satisfy $r - s = 2\mathbb{Z} + 1$.

**Proof of (a):** See [1] (see also [20]).

**Proof of (b):** See Appendix B.

**Remark 1.** Actually, even more is true. The vertex operator algebra $L_{1/2}(c,p',0)$ is both rational and $\sigma$-rational. The rationality was proven by Adamović [1], and later was obtained as a special case of the main result in [20]. The $\sigma$-rationality can be deduced again from Theorem 3.2 (b) and [20], where relative Ext-groups for $SM(p,p')$–minimal models were computed. More precisely

$$\text{Ext}^1_{Vir_c,Vir_0}(L_{\epsilon}(c,p', h_{p,p'}^{r,s}), L_{\epsilon}(c,p', h_{p,p'}^{r',s'})) = 0$$

for all $r, r', s, s'$.

4. Null vectors and $N = 1$ minimal models

The Poincaré-Birhoff-Witt theorem and Theorem 1.1 imply that for a spanning set of the vertex operator superalgebra $V_{1/2}(c,p',0)$ we may choose either

$$L(-i_1) \cdots L(-i_j)G(-k_1) \cdots G(-k_l)1,$$

or

$$L[-i_1] \cdots L[-i_j]G[-k_1] \cdots G[-k_l]1,$$

where $i_1, \ldots, i_j \in \mathbb{N} + 2$ and $k_1, \ldots, k_l \in \mathbb{N} + \frac{3}{2}$. For our purposes it will be more convenient to work with Zhu’s generators $L[n]$ and $G[m]$ (see the formula (1.11)).
Suppose first that \( p \) and \( p' \) are both odd. It is known [3], [20], [21], [22] that there is a singular vector \( v_{p,p',\text{sing}} \in V_{1/2}(c_{p,p'},0) \), which generates the maximal submodule of \( V_{1/2}(c_{p,p'},0) \), uniquely determined by the normalization condition

\[
(4.20) \quad v_{c_{p,p',\text{sing}}} = L[-2]|c_{p,p'}| \mathbf{1} + a_{p,p'} L[-2]|c_{p,p'}|^{-2} G[-5/2] G[-3/2] \mathbf{1} + \cdots ,
\]

where \( a_{p,p'} \) is a constant and the dots denote the "lower" order terms. If \( p \) and \( p' \) are even we have a slightly different situation. There is an odd singular vector \( v_{c_{p,p',\text{sing}}} \) of degree \( \frac{(p-1)(p'-1)}{4} \in \mathbb{N} + \frac{1}{2} \) which generates the maximal submodule. Moreover, we can choose \( v_{c_{p,p',\text{sing}}} \) such that,

\[
(4.21) \quad G[-1/2] v_{c_{p,p',\text{sing}}} = L[-2]|c_{p,p'}| \mathbf{1} + b_{p,p'} L[-2]|c_{p,p'}|^{-2} G[-5/2] G[-3/2] \mathbf{1} + \cdots .
\]

We shall also need the following fact, which is just a consequence of the previous discussion.

**Proposition 4.1.** Let \( L_{1/2}(c_{p,p'}, h_{p,p'}^{r,s}) \) be an irreducible \( L_{1/2}(c_{p,q},0) \)-module. Then

\[
(4.22) \quad Y(v_{\text{sing},c_{p,q}}, x)|_{L_{1/2}(c_{p,p'}, h_{p,p'}^{r,s})} \equiv 0,
\]

\[
(4.23) \quad Y(G[-1/2] v_{\text{sing},c_{p,q}}, x)|_{L_{1/2}(c_{p,p'}, h_{p,p'}^{r,s})} \equiv 0,
\]

where \( L_{1/2}(c_{p,p'}, h_{p,p'}^{r,s}) \) is viewed as a \( V_{1/2}(c_{p,q},0) \)-module.

It is possible to show that the converse is true, i.e., if \( L \) is a \( V_{1/2}(c_{p,q},0) \)-module that satisfies (4.22) and (4.23), then \( L \) is also an \( L_{1/2}(c_{p,q},0) \)-module. This fact follows from the Jacobi identity.

5. **Differential equations and modular forms**

The main result of this section is Theorem 5.1, which should be viewed as a super analogue of the main result from [35]. For simplicity, we will denote the characters of irreducible \( L_{1/2}(c_{p,p'},0) \)-modules by

\[
\text{ch}_1(\tau), \ldots, \text{ch}_{|c_{p,p'}|}(\tau).
\]

The ordering and even a possible normalization of characters will not be of any relevance here. These characters are linearly independent, which can be viewed directly from the leading terms in the \( q \)-expansion of \( \text{ch}_i \). Hence

\[
(5.24) \quad d_{L_{1/2}(c_{p,p'},0)} = |c_{p,p'}|,
\]

where \( d_{LY} \) was defined in Section 1 and \( |c_{p,p'}| \) was defined in Theorem 0.1.

**Theorem 5.1.** (i) Let \( p, p' \in \mathbb{N} \) satisfies the condition (2.13), then characters of irreducible \( L_{1/2}(c_{p,p'},0) \)-modules form a fundamental system of solutions of a differential equation of the form

\[
\left( q \frac{d}{dq} \right)^{|c_{p,p'}|} y - \left\{ 2|c_{p,p'}|(|c_{p,p'}| - 1)E_2(\tau) + \lambda_{p,p'} E_{2,0}(\tau) \right\} \left( q \frac{d}{dq} \right)^{|c_{p,p'}| - 1} y + \cdots + F_{k,0}(\tau) y = 0,
\]

where

\[
(5.25) \quad \lambda_{p,p'} = \begin{cases} 
3|c_{p,p'}|, & p, p' \in 2\mathbb{N} \\
-|c_{p,p'}| + 1, & p, p' \in 2\mathbb{N} + 1,
\end{cases}
\]

and \( F_{i,0}(\tau) \) are certain polynomials in Eisenstein series \( E_2(\tau) \) and \( E_{2l,0}(\tau) \).
(ii) Similarly, the set of characters of irreducible $\sigma$-twisted $L_{1/2}(c_{p,p'},0)$-modules forms a fundamental system of solutions for
\[
\left( q \frac{d}{dq} \right) \left| c_{p,p'} \right| y - \{2|c_{p,p'}|(c_{p,p'} - 1)E_2(\tau) + \lambda_{p,p'}E_{2,1}(\tau)\} \left( q \frac{d}{dq} \right) \left| c_{p,p'} \right| - 1 y + \ldots + F_{k,1}(\tau)y = 0,
\]
where $\lambda_{p,p'}$ is as in (5.25), and $F_{i,1}(\tau)$ are polynomials in Eisenstein series $E_{2i,0}(\tau)$ and $E_{2i,1}(\tau)$.

(iii) Finally, the supercharacters of irreducible $L_{1/2}(c_{p,p'},0)$-modules form a fundamental system for
\[
\left( q \frac{d}{dq} \right) \left| c_{p,p'} \right| y - \{2|c_{p,p'}|(c_{p,p'} - 1)E_2(\tau) + \lambda_{p,p'}E_{2,2}(\tau)\} \left( q \frac{d}{dq} \right) \left| c_{p,p'} \right| - 1 y + \ldots + F_{k,2}(\tau)y = 0,
\]
where $\lambda_{p,p'}$ is as in (5.25), and $F_{i,2}(\tau)$ are certain polynomials in Eisenstein series $E_{2i}(\tau)$ and $E_{2i,2}(\tau)$.

Proof: We will prove part (i). Parts (ii) and (iii) follow along the same lines, with appropriate references to Appendix A. In fact (ii) and (iii) can be obtained directly from (i) if we use the transformation formulas for the characters (for explicit formulas see [21]). We should say that the proof of (i) is similar to our main result in [35], thus we will skip many unnecessary details.

Suppose first that $p$ and $p'$ are odd. Let $M$ be an $L_{1/2}(c_{p,p'},0)$-module and
\[
o(v) := v_{\deg - 1} \in \text{End}(M),
\]
where $v$ is a homogenous vector in $L_{1/2}(c_{p,p'},0)$. From the general form of $v_{c_{p,p'},\text{sing}}$, which is in this case even, Proposition 4.1 and Propositions 8.1, 8.2, 8.3 from Appendix A, it follows that
\[
\text{tr}|_M o(v_{c_{p,p'},\text{sing}})q^{L(0)-c/24} = 0,
\]
can be rewritten as
\[
\sum_{i=0}^{\left| c_{p,p'} \right|} a_i|c_{p,p'}|^{-i}(\tau) \left( q \frac{d}{dq} \right)^i \text{tr}|_M q^{L(0)-c/24} = 0,
\]
where every $a_i(\tau)$ has $q$-expansion with rational powers of $q$. From the previous section we also know that
\[
v_{c_{p,p'},\text{sing}} = L[-2]|c_{p,p'}|1 + \lambda_{p,p'}L[-2]|c_{p,p'}|^{-2}G[-5/2]G[-3/2]1 + \ldots,
\]
so that $a_0(\tau) = 1$. The crucial observation here is that only the first two displayed terms in (5.27) contribute to $a_1(\tau)$. Now, from [34] [35] we know that $L|c_{p,p'}|[-2]1$ contributes with $k(k-1)G_2(\tau)$ to $a_1(\tau)$. Here
\[
G_2(\tau) = -2E_2(\tau) = -\frac{1}{12} + 2q + \ldots,
\]
where $E_2(\tau)$ is the second Eisenstein series. It is a little bit harder to determine the contribution stemming from
\[
\lambda_{p,p'}L[-2]|c_{p,p'}|^{-2}G[-5/2]G[-3/2]1,
\]
simply because we do not have an explicit formula for \( \lambda_{p,p'} \). Nevertheless, we may use Theorem 3.2 and (5.24) to argue that \( \text{ch}_1(\tau), \text{ch}_2(\tau), \ldots, \text{ch}_{|c_{p,p'}|}(\tau) \) form a fundamental system for (5.26). By knowing this much, we are now able to determine at least the leading coefficients in the \( q \)-expansion of \( a_1(\tau) \). Let

\[
\text{ch}_i(\tau) = q^r_i \sum_{n=0}^{\infty} a_n^{(i)} q^n, \quad \text{where} \quad r_i = h_i - c_{p,p'}/24, \quad i = 1, \ldots, |c_{p,p'}|,
\]

so for \( q \to 0 \), asymptotically \( \text{ch}_i(\tau) \to q^{r_i} \). Now, it follows that

\[
(5.28) \quad \sum_{i=1}^{|c_{p,p'}|} r_i = -\text{Coeff}_{q^0}(a_1(\tau))
\]

where \( \text{Coeff}_{q^0}(f(\tau)) \) stands for the constant term in the \( q \)-expansion of \( f \). This can also be seen from the Abel’s lemma (see below). Now, we have to analyze

\[
(5.29) \quad \text{tr}|_{M^0}(L[-2]^k G[-5/2] G[-3/2] 1)q^{L(0)-c/24},
\]

where \( k = |c_{p,p'}| - 2 \). By repeatedly applying Propositions 8.1 and 8.3, the graded trace in (5.29) can be written as

\[
\sum_{j=0}^{k+1} b_{k+1-j}(\tau) \left( q \frac{d}{dq} \right)^j \text{tr}|_{M^0} q^{L(0)-c/24}.
\]

Now, we compute \( b_0(\tau) \) which in turn will give us an explicit formula for \( a_1(\tau) \). The \( L[-2]^k \) factor in (5.29) contributes with the maximal number of \( \left( q \frac{d}{dq} \right)^k \) derivatives. Thus we are left with the computation of

\[
\text{tr}|_{M^0}(G[-5/2] G[-3/2] 1)q^{L(0)-c/24} = 2G_{2,0}(\tau) \left( q \frac{d}{dq} \right) \text{tr}|_{M^0} q^{L(0)-c/24} + c_1(\tau) \text{tr}|_{M^0} q^{L(0)-c/24},
\]

where the right-hand side is derived via Proposition 8.3, by letting \( u = v = G[-3/2] 1 \), together with the formula

\[
\]

From

\[
G_{2,0}(\tau) = 2E_{2,0}(\tau),
\]

we get

\[
a_1(\tau) = -2|c_{p,p'}|(|c_{p,p'}| - 1)E_{2}(\tau) - \lambda_{p,p'} E_{2,0}(\tau),
\]

for some \( \lambda_{p,p'} \). The constant \( \lambda_{p,p'} \) is now computed by using (5.28) and (2.16).

If \( p \) and \( p' \) are even, pick \( G[-1/2] v_{c_{p,p'},\text{sing}} \) instead of \( v_{c_{p,p'},\text{sing}} \) and repeat everything as the above. Finally, apply (2.17). The proof now follows. 

\[\text{3}\] Perhaps this can be determined by using the results from [21].
6. PROOF OF THE MAIN RESULT

Proof of Theorem 0.1: Follows from Theorem 5.1. The only remaining fact is an application of the Abel’s formula for the Wronskian which states that for every basis of solution $y_1, \ldots, y_k$ of a differential equation

$$
\left( q \frac{d}{dq} \right)^k y + a_1(\tau) \left( q \frac{d}{dq} \right)^{k-1} y + \cdots + a_k(\tau)y = 0,
$$

the Wronskian satisfies

$$W\left( q \frac{d}{dq} \right)(y_1, \ldots, y_k) = W(\tau_0)\exp\left( -\int_{\tau_0}^{\tau} a_1(\tau)d(2\pi i\tau) \right).$$

where $\tau_0$ is an arbitrary point in the upper half-plane. Now, both Theorem 0.1 and 0.3 follow that from Theorem 5.1 and Lemma 3 in [34] and the formulas for logarithmic derivatives of $\eta(\tau)$, $f(\tau)$, $f_1(\tau)$ and $f_2(\tau)$ given in the introduction.

Theorem 0.3 follows along the same lines. Similarly we have an analogous result for supercharacters.

Theorem 6.1. For $p, p' \in 2\mathbb{N}$,

$$W_{L_1/2(c_{p,p'},0)}^F(\tau) = \eta(\tau)^{2|c_{p,p'}|(|c_{p,p'}|-1)} f_2(\tau)^{|c_{p,p'}|-1}.$$  

For $p, p' \in 2\mathbb{N} + 1$,

$$W_{L_1/2(c_{p,p'},0)}^F(\tau) = \eta(\tau)^{2|c_{p,p'}|(|c_{p,p'}|-1)} f_2(\tau)^3|c_{p,p'}|.$$  

Remark 2. It is not surprising that three Weber modular functions appear in three different graded traces of $N = 1$ vertex operator superalgebras. If one recalls the formulas for modular transformation of characters in NS and R sector and supercharacters (see for instance [20]), then (6.1) is just a consequence of Theorem 0.1. Similarly, Theorem 0.3 would follows from Theorem 0.1. Thus we really do not require Theorem 3.2 (b) in order to prove Theorem 5.1 (b).

Proof of Proposition 0.5: By utilizing the theory of automorphic forms we will prove a slightly stronger result: For every nonzero $k \in \mathbb{Z}$,

$$W_k(\tau) := W_{(q \frac{d}{dq})^k}((f(\tau))^k, f_1(\tau)^k, f_2(\tau)^k) = \eta(\tau)^{12}.$$  

This, in particular, gives also a new proof of the formula(0.8).

From modular transformation formulas for modular Weber functions (see for instance [41]) it follows that the vector space spanned by $f^k$, $f_1^k$ and $f_2^k$ is modular invariant for every $k$. From the properties of the Wronskian determinants associated to modular invariant spaces and we see that

$$W_k(-1/\tau) = \pm \tau^6 W_k(\tau) \quad \text{and} \quad W_k(\tau + 1) = \pm W_k(\tau),$$

where the sign depends on $k$. Thus $W_k(\tau)^2$ is a holomorphic modular form for $SL(2, \mathbb{Z})$. The order of vanishing of $W_k(\tau)$ at $i\infty$ is $\frac{1}{2}$ for every $k$. Thus $W_k(\tau)^2 = \Delta(\tau)$, which is the 24th power of the Dedekind $\eta$-function. The proof follows.
7. $\mathcal{MS}(2, 4k)$ Superconformal Models

In parallel with $\mathcal{M}(2, 2k + 1)$ Virasoro minimal models, the character formulas of $\mathcal{SM}(2, 4k)$ $N = 1$ minimal models can be simplified. From Proposition 2.2 we get (cf. [6]):

**Lemma 7.1.** For $\epsilon \in \{0, \frac{1}{2}\}$, we have

\[
\text{tr}_{L_p(c_{2,4k}, h_{2,4k}^1 - 2\epsilon)} q^{L(0) - c_{p,p'/24}} = q^{(h_{2,4k}^1 - 2\epsilon - c_{p,p'/24})/24} (-q^{1-\epsilon})^\infty \sum_{j \in \mathbb{Z}} (-1)^j q^{kj^2 + j(k'+1/2-\epsilon)}
\]

where $k' = 0, \ldots, k - 1$.

For convenience, let $s = 2(k - k') - 2\epsilon$. Thus $s = 1, 3, \ldots, 2k - 1$ for $\epsilon = \frac{1}{2}$ and $s = 2, 4, \ldots, 2k$ for $\epsilon = 0$.

Let us recall that

\[
W(q^{\frac{d}{d q}}(f_1, \ldots, f_k)) = f_k W(q^{\frac{d}{d q}}(f_1, \ldots, f_k)).
\]

From the previous lemma, (7.31) and Theorem 5.1 it follows:

**Theorem 7.2.**

\[
\sum_{(n_0, \ldots, n_{k-1}) \in \mathbb{Z}^k} (-1)^{\sum_{i=0}^{k-1} n_i} \prod_{0 \leq i < j \leq k-1} (a(n_i) - a(n_j)) q^{\sum_{i=0}^{k-1} a(n_i)} = C_k \eta(\tau)^{2k-1} \tilde{f}(\tau)^{2k-1},
\]

where $C_k$ is a constant,

\[
\tilde{f}(\tau) = \begin{cases} f(\tau) & \text{for } \epsilon = \frac{1}{2}, \\ f_1(\tau) & \text{for } \epsilon = 0 \end{cases}
\]

and

\[
a(n_i) = h_{2,4k}^1 - 2\epsilon - c_{2,4k} - \frac{1}{24} + \frac{1}{24} \epsilon + kn_i^2 + n_i(i + 1/2 - \epsilon).
\]

**Proof:** Let us fix $k$ and $\epsilon$. For minimal models considered here both $p$ and $p'$ are even. From Theorem 0.1 and 0.3 we know that

\[
W(q^{\frac{d}{d q}}(\text{ch}_1(\tau), \ldots, \text{ch}_k(\tau))) = \frac{\eta(\tau)^{2k-1}}{\tilde{f}(\tau)^{k-1}}
\]

where $k' = 0, 1, 2, \ldots, k - 1$. Now, we apply the formula (7.31) and expand the determinant as in [35] so we get (7.32).

**Remark 3.** The series of $q$-identities in (7.32) are in fact certain specialized Macdonald’s identity for the root system of type $B_l$ [29].
Now, we will examine the case \((2, 8)\) more closely. Before that notice that, by using the Jacobi Triple Product Identity, we get (see for instance [31]):

\[
\text{tr}|_{L_{1/2}(c_{2,4k}, h_{2,4k})} q^{L(0) - c_{2,4k}/24} = q^{1/2 - c_{2,4k}/24} \prod_{n=1}^{\infty} \frac{1}{1 - q^{n/2}}, \quad s = 1, 3, \ldots, 2k - 1,
\]

\[
\text{tr}|_{L_{0}(c_{2,4k}, h_{2,4k})} q^{L(0) - c_{2,4k}/24} = q^{h_{2,4k} - c_{2,4k}/24} \prod_{n=1, n \neq 0 \pm k'(\text{mod} 4)}^{\infty} \frac{1}{1 - q^{2n-1}} \prod_{n=1, n \neq 0 \pm k' \text{(mod} 2k) \neq 0}^{\infty} \frac{1}{1 - q^{n}}, \quad s = 2, 4, \ldots, 2k - 2,
\]

\[
\text{tr}|_{L_{0}(c_{2,4k}, h_{2,4k})} q^{L(0) - c_{2,4k}/24} = q^{h_{2,4k} - c_{2,4k}/24} \prod_{n=1, n \neq 0 \pm k' \text{(mod} 2k) \neq 0}^{\infty} \frac{1 + q^{n}}{1 - q^{n}}.
\]

**Proof of Corollary 0.2.** We apply the previous formulas in the case of \((2, 8)\) and \(\epsilon = 1/2\). The central charge is \(-24\) and \(h_{1,1} = 0, h_{1,3} = -\frac{1}{4}\). The corresponding characters are

\[
\text{ch}_{L_{1/2}(-21/4, 0)}(\tau) = q^{7/32} \prod_{n=1, n \neq 0, \pm 1, \pm 2(\text{mod} 8)}^{\infty} \frac{1}{1 - q^{n}},
\]

\[
= q^{7/32} \prod_{n=0}^{\infty} \frac{1}{1 - q^{(8n+3)/2}}(1 - q^{(8n+4)/2})(1 - q^{(8n+5)/2})^{-1},
\]

and

\[
\text{ch}_{L_{1/2}(-21/4, -1/4)}(\tau) = q^{-1/32} \prod_{n=0}^{\infty} \frac{1}{1 - q^{(8n+1)/2}}(1 - q^{(8n+4)/2})(1 - q^{(8n+7)/2})^{-1}.
\]

Theorem 0.1 implies

\[
\mathcal{W}_{L_{1/2}(-21/4, 0)}(\tau) = \frac{\eta(\tau)^4}{\eta(\tau/2)^2} = \eta(\tau)^2 \eta(\tau/2) \eta(2\tau),
\]

where in the last equation we used

\[
\frac{\eta(\tau)}{\eta(\tau/2)} = \frac{\eta(\tau/2)^2}{\eta(2\tau)}.
\]

Now, after some computations the Wronskian can be rewritten as

\[
\left(1 - 2 \sum_{n=0}^{\infty} \frac{(8n + 1)q^{(8n+1)/2}}{1 - q^{(8n+1)/2}} \right) + \left(8n + 7 \right) q^{(8n+7)/2} - \frac{(8n + 3)q^{(8n+3)/2}}{1 - q^{(8n+3)/2}} - \frac{(8n + 5)q^{(8n+5)/2}}{1 - q^{(8n+5)/2}}.
\]

(7.33) \(\text{ch}_{L_{1/2}(-21/4, 0)}(\tau) \text{ch}_{L_{1/2}(-21/4, -1/4)}(\tau) = \frac{\eta(\tau)^4}{\eta(\tau/2)^2} \eta(2\tau)\).

If we substitute \(\tau\) by \(2\tau\) and use

\[
\text{ch}_{L_{1/2}(-21/4, 0)}(2\tau) \text{ch}_{L_{1/2}(-21/4, -1/4)}(2\tau) = q^{3/8} \prod_{n=1}^{\infty} \frac{1 - q^{2n}(1 - q^{8n})^2}{(1 - q^{4n})^2(1 - q^{8n})} = \frac{\eta(2\tau)\eta(8\tau)^2}{\eta(4\tau)^2\eta(\tau)^2}.
\]
then
\[
\mathcal{W}_{L_{1/2}}^{NS}(-21/4,0)(2\tau) = \eta(2\tau)^2 \eta(\tau)\eta(4\tau)
\]
and (7.33) yield the Carlitz’s formula.

**Proof of Corollary 0.4.** Here, \(\epsilon = 0\), \(c = -\frac{21}{4}\),
\[
\text{ch}_{L_0}(-21/4,-3/32)(\tau) = q^{1/8} \prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^{2n+2}} = q^{1/8} \prod_{n=1}^{\infty} (1 + q^n)(1 + q^{2n})
\]
and
\[
\text{ch}_{L_0}(-21/4,-7/32)(\tau) = \prod_{n=1}^{\infty} \frac{1 + q^{2n-1}}{1 - q^{2n-1}} = \prod_{n=1}^{\infty} (1 + q^{2n-1})(1 + q^n).
\]
Now, Theorem 0.3 implies
\[
\mathcal{W}_{L_{-21/4}}^{R}(-21/4,0)(\tau) = \frac{\eta(\tau)}{\eta_1(\tau)} = q^{1/8} \prod_{n=1}^{\infty} \frac{1 - q^n}{1 + q^n}
\]
From these formulas and the following Gauss’ identity
\[
\sum_{n \in \mathbb{Z}} (-q)^n = \prod_{n=1}^{\infty} \frac{1 - q^n}{1 + q^n},
\]
we get (0.4).

**Remark 4.** Originally, we wanted to understand some of the formulas in [24], [36] in connection with [34] and [35]. Our hope was to obtain formulas for sums of squares from certain ”small” rational vertex operator superalgebras (e.g., \(N = 1\) minimal models). Our Theorem 0.3 implies that formulas for sums of squares can be obtained directly if only if
\[
2|c_{p,p'}|(|c_{p,p'}| - 1) = |c_{p,p'}| - 1,
\]
which holds only in the trivial case \(|c_{p,p'}| = 0\) or 1. Thus, we should not expect any formula for sums of squares. This turns out not to be completely true because for \(SM(2,8)\) minimal models we obtained Jacobi Four Square Theorem after some minor manipulations with \(q\)-series. It is an open problem to find examples of \(N = 1\) vertex operator superalgebras that would give formulas for sums of squares by using the ideas from this paper.

8. **Appendix A**

In this part we recall and prove some recursion formulas for graded traces needed in untwisted and twisted sector.
Let us introduce the following series

\[(8.34)\]  
\[P_{m+1}(x, q) = \frac{1}{m!} \left( \sum_{n \geq 1} \frac{n^m x^n}{1 - q^n} + (-1)^{m+1} \frac{n^m x^{-n} q^n}{1 - q^n} \right),\]

\[P_{m+1,1}(x, q) = \frac{1}{m!} \left( \sum_{n \geq 1} \frac{(n - 1/2)^m x^{n-1/2}}{1 + q^{n-1/2}} + (-1)^m \frac{(n - 1/2)^{m-1/2} x^{-n+1/2} q^{n-1/2}}{1 + q^{n-1/2}} \right),\]

\[P_{m+1,0}(x, q) = \frac{1}{m!} \left( \sum_{n \geq 1} \frac{n^m x^n}{1 + q^n} + (-1)^m \frac{n^m x^{-n} q^n}{1 + q^n} \right),\]

\[P_{m+1,1}^{\mathcal{S}}(x, q) = \frac{1}{m!} \left( \sum_{n \geq 1} \frac{(n - 1/2)^m x^{n-1/2}}{1 - q^{n-1/2}} + (-1)^{m+1} \frac{(n - 1/2)^{m-1/2} x^{-n+1/2} q^{n-1/2}}{1 - q^{n-1/2}} \right).\]

The following proposition is essentially from [42] (see also [33] and [11]).

**Proposition 8.1.** Let \( V \) be a vertex operator algebra (resp. superalgebra), \( M \) a \( V \)-module and \( u, v \in V \) two vectors (resp. \( u, v \in V^0 \) two even vectors), then

(i) \[\text{tr}|_M X(u, x_1)X(v, x_2)q^{L(0)-c/24} = \text{tr}|_M o(u)o(v)q^{L(0)-c/24} + \sum_{i \geq 0} P_{i+1} \left( \frac{x_2}{x_1}, q \right) \text{tr}|_M o(u[i]v)q^{L(0)-c/24}.\]

(ii) \[\text{tr}|_M X(Y[u, y]v, x)q^{L(0)-c/24} = \text{tr}|_M o(u)o(v)q^{L(0)-c/24} + \sum_{m \geq 0} \varphi_{m+1}(y, \tau) \text{tr}|_M o(u[m]v)q^{L(0)-c/24},\]

where \[\varphi_m(y, \tau) = \frac{1}{ym} + (-1)^m \sum_{k \geq m/2} \frac{\binom{2k-1}{m-1}}{2k} G_{2k}(\tau) y^{2k-m}.\]

Now, we prove a "super" analogue of Proposition 8.1.

**Proposition 8.2.** Let \( V \) be a vertex operator superalgebra, \( M \) a \( V \)-module, and \( u, v \in V^1 \) (odd vectors). We have

\[(8.35)\]  
\[\text{tr}|_M X(u, x_1)X(v, x_2)q^{L(0)-c/24} = \sum_{m \geq 0} P_{m+1,1/2} \left( \frac{x_2}{x_1}, q \right) X(u[m]v, x_2)q^{L(0)-c/24}.\]

**Proof:** We will use the following formula from [33]:

\[\left[ X(u, x_1), X(v, x_2) \right]_+ = \text{Res}_y \delta_{1/2} \left( \frac{eyx_2}{x_1} \right) X(Y[u, y]v, x_2).\]
Now from the properties of the trace functional
\[
\text{tr}|_M (1 + q^{-Dx_1}) X(u, x_1) X(v, x_2) q^{L(0) - c/24} =
\]
\[
(8.36) \quad \text{tr}|_M \sum_{m \geq 0} \frac{D^m}{m!} \delta_{1/2} \left( \frac{x_2}{x_1} \right) X(u[m]v, x_2) q^{L(0) - c/24}.
\]
By comparing the coefficients of the \( x_1 \) and \( x_2 \) and using the delta function expansion we get the formula.

Next we introduced (normalized) Eisenstein series for \( \Gamma(1) \):
\[
G_{2k}(\tau) = -\frac{B_{2k}}{(2k)!} + \frac{2}{(2k - 1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1 - q^n},
\]
and certain linear combination of level 2 Eisenstein series
\[
G_{2k,0}(\tau) = \frac{B_{2k}(1/2)}{(2k)!} + \frac{2}{(2k - 1)!} \sum_{n=1}^{\infty} \frac{(n - 1/2)^{2k-1} q^{n-1/2}}{1 + q^{n-1/2}},
\]
where \( B_{2k}(x) \) is the Bernoulli polynomial. From the duplication formula
\[
B_{2k}(1/2) = -(1 - 2^{1-2k}) B_{2k}
\]
we have
\[
G_{2,0}(\tau) = -\frac{1}{24} + 2 \sum_{n=1}^{\infty} \frac{(n - 1/2) q^{n-1/2}}{1 + q^{n-1/2}}.
\]
Notice that, \( G_{2}(\tau) \) is a multiple of \( E_2(\tau) \) and \( G_{2,0}(\tau) \) is a multiple of \( E_{2,0}(\tau) \) (see the Introduction).

**Proposition 8.3.** Let \( V, M, u \) and \( v \) be as in Proposition 8.2. Then,
\[
\text{tr}|_M X(Y[u, y]v, x) q^{L(0) - c/24} = \sum_{m \geq 0} \varphi_{m+1,1/2}(y, \tau) X(u[m]v, x) q^{L(0) - c/24},
\]
where
\[
\varphi_{m,1/2}(y, \tau) = \frac{1}{y^m} + (-1)^{m-1} \sum_{k \geq m/2} \binom{2k - 1}{m - 1} G_{2k,0}(\tau) y^{2k-m}.
\]

**Proof:** Similar results were obtained in [33] [11] and [42], so we skip all unnecessary explanations.

By using Proposition (8.2) we have (cf. [33])
\[
\text{tr}|_M X(Y[u, y]v, x) q^{L(0) - c/24} = \sum_{m \geq 1} \left\{ \text{Res}_t \left( \frac{e^{y \deg(u) t^{-\deg(u)}}}{1 - e^t / t} P_{m,1/2}(t^{-1}, q) - t^{-\deg(u)} e^{y \deg(u) - 1} \frac{1}{1 - t / e^y} P_{m,1/2}(qt^{-1}, q) \right) \right. \]
\[
+ \text{Coeff}_{x \geq m-1} \left( \frac{e^{(y - x) \deg(u)}}{e^y - x - 1} \right) \right\} \text{tr}|_M o(u[m-1]v) q^{L(0) - c/24}.
\]
\[
(8.37)
\]
Now,
\[
\text{Rest} \left\{ e^{y\deg(a)} t^{-\deg(u)} \frac{1}{1 - e^y/t} P_{m,1/2}(t^{-1}, q) - t^{-\deg(u)} e^{y\deg(u) - 1} \frac{1}{1 - t/e^y} P_{m,1/2}(qt^{-1}, q) \right\}
\]
\[
= (-1)^{m-1} \sum_{n \in \mathbb{N}} e^{y(n\deg(u) + n)} q^{(\deg(u) + n)m - 1} \frac{1}{1 + q^{(\deg(u) + n)}}
\]
\[
- \left\{ \frac{1}{(m-1)!} \sum_{n \in \mathbb{N}+1/2} e^{-y} n^{m-1} q^n \frac{1}{1 + q^n} + (-1)^{m-1} \frac{(m-1)!}{(m-1)!} \sum_{n \geq 1/2, n \in \mathbb{N}+1/2} \frac{e^{y} n^{m-1}}{1 + q^n} \right\}
\]
\[
= (-1)^{m-1} \sum_{n \in \mathbb{N}} e^{y(n\deg(u) + n)} q^{(\deg(u) + n)m - 1} \frac{1}{1 + q^{(\deg(u) + n)}} - \frac{1}{(m-1)!} \sum_{n \in \mathbb{N}+1/2} e^{-y} n^{m-1} q^n \frac{1}{1 + q^n}
\]
\[
+ \frac{(-1)^m}{(m-1)!} \left\{ \sum_{n \geq 1/2, n \in \mathbb{N}+1/2} \frac{e^{y} n^{m-1} q^n}{1 + q^n} \right\} - \sum_{n \geq 1/2, n \in \mathbb{N}+1/2} \frac{e^{-y} n^{m-1} q^n}{1 + q^n}
\]
(8.38)
\[
= (-1)^{m-1} \sum_{n \in \mathbb{N}} \left\{ e^{y} n^{m-1} q^n \frac{1}{1 + q^n} + (-1)^{m} e^{-y} n^{m-1} q^n \right\} + \frac{(-1)^m}{(m-1)!} \sum_{n \geq 1/2, n \in \mathbb{N}+1/2} e^{y} n^{m-1}.
\]

By using a well-known formula
\[
\sum_{n=0}^{N-1} (a + \lambda)^{k-1} = \frac{B_k(N) - B_k(\lambda)}{k},
\]
where $B_k(x)$ is the $k$-th Bernoulli polynomial, we have the following identity
(8.39)
\[
\sum_{n \geq 1/2, n \in \mathbb{N}+1/2} e^{y} n^{m-1} q^n = \sum_{n=0}^{\infty} \frac{y^n (B_{m+n}(\deg(u)) - B_{m+n}(1/2))}{(n+m)!}.
\]

Now,
\[
\text{Coeff}_{x^{m-1}} e^{(y-x)\deg(a)} e^{y-x} = (-1)^{m-1} \frac{(\partial}{\partial y})^m e^{\deg(a)y} \frac{1}{e^y - 1}
\]
(8.40)
\[
= (-1)^{m-1} \frac{(\partial}{\partial y})^m \frac{1}{(m-1)!} \sum_{n=0}^{\infty} \frac{B_n(\deg(u)) y^{n-1}}{n!} = \frac{1}{y^m} + (-1)^{m-1} \frac{1}{(m-1)!} \sum_{n \geq 0} \frac{B_{m+n}(\deg(u)) y^n}{(n+m)!}.
\]

By combining (8.39) and (8.40) with (8.38) and (8.37) we get
\[
\text{tr}_{|M} X(Y\{u,v,x\}) q^{L(0)-c/24}
\]
\[
= \sum_{m \geq 1} \left( (-1)^{m-1} \frac{1}{(m-1)!} \sum_{n \in \mathbb{N}+1/2} \left\{ e^{y} n^{m-1} q^n \frac{1}{1 + q^n} + (-1)^{m} e^{-y} n^{m-1} q^n \right\} \right) \text{tr}_{|MO}(u[m-1]v) q^{L(0)-c/24}
\]
(8.41)
\[
+ \frac{1}{y^m} + \frac{1}{(m-1)!} \sum_{n=0}^{\infty} \frac{B_{m+n}(1/2) y^n}{(n+m)!} \right) \text{tr}_{|MO}(u[m-1]v) q^{L(0)-c/24}.
\]

Finally, for $m \geq 1$,
\[
\frac{1}{y^m} + \frac{(-1)^{m-1}}{(m-1)!} \sum_{n=0}^{\infty} \frac{B_{m+n}(1/2)y^n}{(n+m)!} + \frac{(-1)^m e^{yn} n^{m-1} q^n}{1+q^n}
\]
\[= \frac{1}{y^m} + \frac{(-1)^{m-1}}{(m-1)!} \sum_{n=0}^{\infty} \frac{B_{m+n}(1/2)y^n}{(n+m)!} + \frac{(-1)^{m-1}}{(m-1)!} \sum_{n \in \mathbb{N}+1/2} \sum_{k \geq 0} \frac{y^k n^{k+m-1} q^n}{k!(1+q^n)} + \frac{(-1)^m k y^k n^{k+m-1} q^n}{k!(1+q^n)}
\]
\[= \frac{1}{y^m} + \frac{(-1)^{m-1}}{(m-1)!} \sum_{n=0}^{\infty} \frac{B_{m+n}(1/2)y^n}{(n+m)!} + \frac{(-1)^{m-1}}{(m-1)!} \sum_{l \geq m/2} \sum_{n \in \mathbb{N}+1/2} \frac{y^{2l-m} n^{2l-1} q^n}{(2l-m)!(1+q^n)}
\]
\[= \frac{1}{y^m} + (-1)^{m-1} \sum_{l \geq m/2} \left( \frac{B_{2l}(1/2)y^{2l-m}}{(2l)!} + 2 \sum_{n \in \mathbb{N}+1/2} \frac{y^{2l-m} n^{2l-1} q^n}{(2l-m)!(1+q^n)} \right)
\]
\[= \frac{1}{y^m} + (-1)^{m-1} \sum_{l \geq m/2} \left( \frac{2l-1}{m-1} y^{2l-m} \left( \frac{B_{2l}(1/2)}{(2l)!} + \frac{2}{(2l-1)!} \sum_{n \in \mathbb{N}+1/2} \frac{n^{2l-1} q^n}{(1+q^n)} \right) \right).
\]

Now we consider a $\sigma$-twisted $V$-module $M^\sigma$. Let $u \in V^1$. Then
\[Y^\sigma(u,x) \in \text{End}(M^\sigma)[x, x^{-1}].\]

Thus
\[X^\sigma(u,x) \in \text{End}(M^\sigma)[[x, x^{-1}]].\]

**Proposition 8.4.** Let $M^\sigma$ be a $\sigma$-twisted $V$-modules and $u,v \in V^1$, then
\[\text{tr}|_{M^\sigma} X^\sigma(u,x_1)X^\sigma(v,x_2)q^{L(0)-c/24}
\]
\[= \text{tr}|_{M^\sigma} \sigma^\sigma(u)\sigma^\sigma(v)q^{L(0)-c/24} + \sum_{m \geq 0} P_{m,0} \left( \frac{x_2}{x_1}, q \right) \text{tr}|_{M^\sigma} X(u[m]v, x_2)q^{L(0)-c/24}.
\]

**Proof:** Follows from
\[[X^\sigma(u,x_1), X^\sigma(v,x_2)] = \text{Res}_y \delta \left( \frac{e^y x_2}{x_1} \right) X^\sigma(Y[u,y]v, x_2)
\]
and arguments similar as in Proposition 8.3.

We introduce
\[G_{2k,1}(\tau) = \frac{B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n \geq 1} \frac{n^{2k-1} q^n}{1+q^n}.
\]

**Proposition 8.5.**
\[\text{tr}|_{M^\sigma} X^\sigma(Y[u,y]v, x)q^{L(0)-c/24}
\]
\[= \text{tr}|_{M^\sigma} \sigma^\sigma(u)\sigma^\sigma(v)q^{L(0)-c/24} + \sum_{m \geq 0} q_{m+1,0}(y, \tau) \text{tr}|_{M^\sigma} X^\sigma(u[m]v, x)q^{L(0)-c/24},
\]
where
\[q_{m,0}(y, \tau) = \frac{1}{y^m} + (-1)^m \sum_{k \geq m/2} \left( \frac{2k-1}{m-1} \right) G_{2k,1}(\tau) y^{2k-m}.
\]
Finally, we define
\[ G_{2k,2}(\tau) = \frac{B_{2k}(1/2)}{(2k)!} - \frac{2}{(2k-1)!} \sum_{n \geq 1} \frac{(n-1/2)^{2k-1}q^{n-1/2}}{1-q^{n-1/2}}. \]

**Proposition 8.6.** For \( u, v \in V^0 \) the Proposition 8.1 holds verbatim with the trace replaced by supertrace. However, for \( u, v \in V^1 \) we have
\[
\text{str}|_M X(Y[u, y]v, x)q^{L(0)-c/24} = \sum_{m \geq 0} \varphi_{m,0}^\sigma(y, \tau) \text{str}|_M X(u[m]v, x)q^{L(0)-c/24},
\]
where
\[
\varphi_{m,0}^\sigma(y, \tau) = \frac{1}{y^m} + (-1)^m \sum_{k \geq m/2} \binom{2k-1}{m-1} G_{2k,2}(\tau) y^{2k-m}.
\]

**9. Appendix B**

In this part we prove Theorem 3.2 (b). Every irreducible \( \sigma \)–twisted module is of the form \( L_0(c_{p,q}, h) \), for some \( h \in \mathbb{C} \) so we have to classify all possible weights \( h \).

We will use a \( \sigma \)–twisted version of the Zhu’s associative algebra for vertex operator superalgebra [40]. The untwisted version was considered in [22]. Consider a subspace \( O(V) \subset V \), spanned by elements of the form
\[ \text{Res}_x \left( 1 + x \right)^{\deg(u)} x^2 Y(u, x)v, \]
where \( u \in V \) is homogeneous. It can be easily shown that
\[ \text{Res}_x \left( 1 + x \right)^{\deg(u)} x^n Y(u, x)v \in O(V), \quad n \geq 2. \]
Then, the vector space \( A_\sigma(V) = V/O(V) \) can be equipped with an associative algebra structure via
\[ u * v = \text{Res}_x \left( 1 + x \right)^{\deg(u)} x Y(u, x)v \]
(see [40]). An important difference between the untwisted associative algebra and \( A_\sigma(V) \) is that \( A_\sigma(V) \) is \( \mathbb{Z}_2 \)–graded, so
\[ A_\sigma(V) = A^0_\sigma(V) \oplus A^1_\sigma(V). \]
Let us describe explicitly the associative algebra \( A_\sigma(V_{1/2}(c_{p,q}, 0)) \). Let \( V = V_{1/2}(c_{p,q}, 0) \). It is not hard to see that the following formulas hold:
\[
(L(-m-2) + 2L(-m-1) + L(-m))v \in O(V), \quad m \geq 2
\]
\[ (L(0) + L(-1))v \in O(V), \]
\[ L(-m)v \equiv (-1)^m((m-1)(L(-2) + L(-1)) + L(0))v \mod O(V), \quad m \geq 2,
\]
\[
\sum_{n \geq 0} \left( \frac{3}{2} \right)^n G(-3/2 - i + n)v \in O(V), \quad i \geq 1,
\]
for every \( v \in V \).
Let us denote by \([ \ ]\) the projection from \(V\) to \(A_\theta(V)\). Also, let
\[
 x = [L(-2)1], \quad \theta = [G(-3/2)1].
\]
Then
\[
 [(L(-2) + L(-1))v] = x \ast [v].
\]
Also
\[
 \theta \ast \theta = \left[ \left( L(-2) - \frac{c}{24} \right) 1 \right] = x - \frac{c}{24},
\]
where \(c \in \mathbb{C}\). Therefore (cf. [22]), as a \(\mathbb{Z}_2\)-graded vector space
\[
 A_\theta(V) \cong \mathbb{C}[x] \oplus \theta \mathbb{C}[x].
\]
Consider the category \(\mathcal{C}\) of \(\mathbb{Z}_2\)-graded \(A_\theta(V)\)–modules. It is clear that every irreducible module in \(\mathcal{C}\) is uniquely determined by the action of \(\theta\).

**Lemma 9.1.** Irreducible modules in \(\mathcal{C}\) are either one-dimensional or two-dimensional. In the former case, \(\theta\) acts as zero and \(x\) acts as the multiplication with \(h = \frac{c}{24}\). In the latter case we can choose a basis in which \(\theta\) acts via
\[
(9.46) \quad \left[ \begin{array}{cc} 0 & h - \frac{c}{24} \\ 1 & 0 \end{array} \right],
\]
and \(x\) acts as the multiplication with \(h \neq \frac{c}{24}\).

**Proof:** Let \(R = R_0 \oplus R_1\) be an irreducible \(A_\theta(V)\)–module. There exists \(v \in R_i\) such that \(x \cdot v = \lambda v\), which makes the vector space spanned by \(v\) and \(\theta v\) a submodule of \(R\). Thus \(R\) is at most two-dimensional and \(x\) acts as a scalar multiplication. If \(\theta\) acts trivially, then \(R\) is one dimensional and in \(\mathcal{C}\) if and only if \(h = \frac{c}{24}\). If \(\theta \neq 0\), then \(R = \text{ Span } \{v, \theta v\}\), \(\theta\) acts via \((9.46)\), and \(x\) acts as the scalar multiplication by \(h\). Clearly, \(h \neq \frac{c}{24}\), otherwise the one dimensional space spanned by \(\theta v\) would be a submodule of \(R\). \(
\]
Let us recall that, according to our definition, a \(\sigma\)-twisted \(L_{1/2}(c,0)\)-module is a \(\mathbb{Z}_2\)-graded module for the \(N = 1\) Ramond superalgebra. The following result is essentially from [40], applied to the \(N = 1\) vertex operator superalgebra \(L_{1/2}(c,0)\).

**Theorem 9.2.** There is a one-to-one correspondence between the equivalence classes of irreducible \(A_\sigma(L_{1/2}(c,0))\)–modules in \(\mathcal{C}\), and the equivalence classes of irreducible \(\sigma\)-twisted \(L_{1/2}(c,0)\)–modules.

We should say here that the correspondence in [40] is only between irreducible \(A_\sigma(V)\)-modules (where \(A_\sigma(V)\) is viewed as an associative algebra) and ungraded \(\sigma\)-twisted \(L_{1/2}(c,0)\)-modules. But in fact, the correspondence carries over to the graded case as well.

Now we pass from \(V_{1/2}(c,0)\) to \(L_{1/2}(c,0)\). Let us recall that inside \(V_{1/2}(c_{p,p'},0)\) there is a vector \(v_{c_{p,p'},\text{sing}}\) of degree \((p-1)(p'-1)\) \(+\) \((-1)^{p' + 1}\) which generates the maximal submodule \(V^{(1)}_{1/2}(c_{p,p'},0) \subset V_{1/2}(c_{p,p'},0)\). Moreover, for \(p \in 2\mathbb{N}\) the generating vector is odd and for \(p \in 2\mathbb{N} + 1\) the generating vector is even. We have the following short exact sequence:
\[
0 \rightarrow V^{(1)}_{1/2}(c,0) \rightarrow V_{1/2}(c,0) \rightarrow L_{1/2}(c,0) \rightarrow 0,
\]
where \( V^{(1)}_{1/2}(c,0) \) is the maximal submodule of \( V_{1/2}(c,0) \). As in [22], [39] it follows that

\[
A_\sigma(L_{1/2}(c,0)) \cong \mathbb{C}[x] \oplus \theta \mathbb{C}[x]/I^c,
\]

where \( I^c \) is an \( A_\sigma(V) \)-ideal. The ideal \( I^c \), viewed as a \( \mathbb{C}[x] \)-module, decomposes as

\[
I^c = I^c_1 \oplus \theta I^c_2,
\]

where

\[
I^c_1 = \langle p^c_1(x) \rangle \quad \text{and} \quad I^c_2 = \langle p^c_2(x) \rangle
\]

are two \( \mathbb{C}[x] \)-ideals, generated by a pair of monic polynomials \( p^c_1(x) \) and \( p^c_2(x) \). The ideal \( I^c \) is generated by one polynomial which is depending on the parity, even or odd. Thus, by knowing a single polynomial \( p^c_1(x) \) or \( p^c_2(x) \) the other monic polynomial is uniquely determined. More precisely

**Lemma 9.3.** Let \( c = c_{p,p'} \). Then,

(i) For \( p,p' \in 2\mathbb{N}, p^c_1(x) = p^c_2(x)(x - c/24) \).

(ii) For \( p,p' \in 2\mathbb{N} + 1, p^c_1(x) = p^c_2(x) \).

**Proof:** For (i), recall that the maximal ideal of \( V_{1/2}(c_{p,p}, 0) \) is generated by an odd vector, so the ideal \( I^c \) is generated by \( \theta p^c_2(x) \). Now, \( \theta \cdot \theta p^c_2(x) = (x - c/24)p^c_1(x) \) is an (even) polynomial of the smallest degree in \( I^c \). Thus, \( p^c_1(x) = p^c_2(x)(x - c/24) \).

For (ii), the maximal ideal is generated by an even vector, with the projection being \( p^c_1(x) \). Clearly, \( I^c_2 \) is also generated by the same monic polynomial. 

**Remark 5.** Notice that for \( p \) being odd

\[
\begin{align*}
[G(-1/2)v_{c_{p,p'},\text{sing}}] &= 0, \\
[G(-3/2)v_{c_{p,p'},\text{sing}}] &= \theta p_1(x),
\end{align*}
\]

while for \( p \) even

\[
[G(-1/2)v_{c_{p,p'},\text{sing}}] = (x - c/24)p_2(x).
\]

**Lemma 9.4.** \( L_0(c,h) \) is an \( L_{1/2}(c,0) \)-module if and only if \( p^c_1(x) \) and \( \theta p^c_2(x) \) act trivially on \( L_0(c,h)_0 \) (i.e., on the top degree subspace of \( L_0(c,h) \)).

**Proof:** Follows straight from the description of \( A_\sigma(L_{1/2}(c,0)) \) and the fact that \( x = [L(-2)1] \) acts as the multiplication with \( h \).

Let us now focus on \( [v_{c_{p,p'},\text{sing}}] \) (resp. \( [G(-1/2)v_{c_{p,p'},\text{sing}}] \), which for \( p \) odd (resp. \( p \) even) give \( p^c_1(x) \). Suppose that \( [w] = [L(-m)v] \), where \( L(0) \cdot w = jw \), then by (9.45)

\[
[L(-m)v] = (-1)^m(m-1)x \ast [v] + (-1)^m(j - m)[v].
\]

Similarly, by using (9.45) and the identity

\[
\begin{align*}
\binom{3/2}{n + 1 + k} &= \sum_{i=0}^{n+1} \binom{3/2}{i} \binom{-3/2}{n-i+1} \binom{3/2}{k} \frac{(2n - 2i + 5)k}{3(n-i+1+k)},
\end{align*}
\]

which holds for every \( k, n \in \mathbb{N} \), we have for every \( n \geq 0 \)

\[
[G(-n - 3/2)v] = \left( \binom{-3/2}{n} \theta \ast [v] - \frac{2n + 3}{3} \sum_{k \geq 1} \binom{3/2}{k} \frac{k}{n+k}[G(k - 3/2)v] \right).
\]
Now, we will be able to classify all irreducible $A_{\sigma}(L_{1/2}(c_{p,p'},0))$–modules just by knowing the explicit formulas for $[v_{c_{p,p'},\text{sing}}]$ and $[G(-1/2)v_{c_{p,p'},\text{sing}}]$. Equivalently, it suffices to show that $p'_1(x)$ acts trivially on a pair of eigenvectors $v^\sigma$, $\sigma = \pm$, with respect to the action of $G(0)$,

$$G(0) \cdot v^\sigma = \sigma \sqrt{h - \frac{c_{p,p'}}{24}} v^\sigma.$$ 

The following result is essentially from [20].

**Lemma 9.5.** Let $p,p'$ be as in (2.13). Then,

$$p'_1(x) = \prod_{h \in H^0_{p,p'}} (x - h),$$

where $H^0_{p,p'}$ was defined in (2.14).

**Proof:** Let us assume that $p$ (and hence $p'$) is odd. For $p$ even (or $p'$ even) exactly the same proof applies with $[v_{c_{p,p'},\text{sing}}]$ replaced by $[G(-1/2)v_{c_{p,p'},\text{sing}}]$.

By repeatedly applying (9.48) and (9.49) we can in theory express $[v_{c_{p,p'},\text{sing}}]$ as a polynomial in $x$. This was done in [20], but in a slightly different setup. Unlike our approach that uses $A_{\sigma}(V)$, Iohara and Koga in [20] obtained polynomials $p'_1(x)$ via certain coinvariant calculations. Our formulas (9.48) and (9.49) are "dual" to those in Lemma 7.3 of [20]. Let us compute the action of (9.48) and (9.49) on the vectors $v^\sigma$:

$$[L(-m)v] \cdot v^\sigma = (-1)^m(m - 1)h[v] \cdot v^\sigma + (-1)^m(j - m)[v] \cdot v^\sigma,$$

$$[G(-n - 3/2)v] \cdot v^\sigma = \left(-\frac{1}{4}\right)^n (2n + 1) \left(\frac{2n}{n}\right) \left(\frac{n}{24}\right)^2 \cdot v^\sigma - \frac{2n + 3}{3} \sum_{k \geq 1} \left(\frac{3/2}{k}\right) \frac{k}{n + k} [G(k - 3/2)v] \cdot v^\sigma,$$

where in the last equation we used the trivial identity:

$$\left(-\frac{3/2}{n}\right) = \left(-\frac{1}{4}\right)^n (2n + 1) \left(\frac{2n}{n}\right).$$

Compared to Lemma 7.3 [20] it appears that our formulas (9.50) differ up to some multiplicative factors. But this discrepancy is corrected (see p. 335 in [20] for details) after an application of the antipode map $a(x) = -(-1)^{\text{deg}(x)} x$, $x \in \text{Vir}_{1/2}$. Now, the computation of $[v_{c_{p,p'},\text{sing}}]$ and $[v_{c_{p,p'},\text{sing}}] \cdot v^\sigma$ reduces to Lemma 7.4 [20], after we let $h_1 = 0$ and $h_0 = h_\infty = h$. Theorem 7.2 and Remark 7.5 in [20] now give the proof.

**Corollary 9.6.** The irreducible $\sigma$–twisted $L_{1/2}(c_{p,p'},0)$–modules are exactly $\mathcal{S}M(p,p')$ minimal models in the Ramond sector.

**Proof:** It follows directly from Lemmas 9.1, 9.3–9.5.
References


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