Permutation orbifolds of rank three fermionic vertex superalgebras

Antun Milas, Michael Penn, and Josh Wauchope

Abstract We describe the structure of the permutation orbifold of the rank three free fermion vertex superalgebra (of central charge $\frac{3}{2}$) and of the rank three symplectic fermion vertex superalgebra (of central charge -6).

Key words: Fermions, Vertex algebras, $\mathcal{W}$-algebras

1 introduction

Invariant subalgebras of free fields vertex algebras and superalgebras are rich sources of interesting simple vertex algebras. There is already a substantial body of work on this subject, especially from the perspective of $\mathcal{W}$-algebras. These approaches are primarily based on application of classical invariant theory (as in [7, 17, 19]). Interesting $\mathcal{W}$-algebras that arise from finite orbifolds show up in the classification of $c = 1$ rational vertex algebras (cf. [9]). Similarly, new examples of C_2-cofinite vertex algebras come from for the triplet vertex algebra [1] and its ADE orbifolds [2, 3, 20].

When it comes to permutation orbifolds (fixed under the *full* symmetric group S_n) very little is known except for $n = 2$. Recently, the first two authors, jointly with Shao, have investigated the structure of the permutation orbfiold of the rank three Heisenberg algebra under the full symmetric group, denoted by $\mathcal{H}(3)^{S_3}$ [21]. They proved that this is a $\mathcal{W}$-algebra of type

Antun Milas
Department of Mathematics and Statistics, SUNY-Albany, 1400 Washington Avenue , e-mail: amilas@albany.edu

Michael Penn
Randolph College, 2500 Rivermont Avenue, e-mail: mpenn@randolphcollege.edui

Josh Wauchope
Department of Mathematics and Statistics, SUNY-Albany, 1400 Washington Avenue

$(1, 2, 3, 4, 5, 6^2)$. In this note we establish similar results for fermionic vertex superalgebras.

Let us recall here

$$\mathcal{F} = \Lambda(\phi(-1/2), \phi(-3/2), ...),$$

the rank one free fermion vertex superalgebra generated by an odd field $\phi(z)$, with super-brackets

$$[\phi(n), \phi(m)]_+ = \delta_{n+m,0}.$$

We consider the S_3-orbifold of three copies of $\mathcal{F}$:

$$(\mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F})^{S_3},$$

with S_3 permuting the tensor factors (with signs). Our first main result, Theorem 1, pertains to the inner structure of this vertex supealgebra. We prove it is of type $(\frac{1}{2}, 2, 4, \frac{9}{2})$. We also obtain a related result for $(\mathcal{F}\otimes\mathcal{F}\otimes\mathcal{F})^{\mathbb{Z}_3}$ under a 3-cycle permutation of tensor factors. We also give a solely bosonic description of $\mathcal{F}(3)^{S_3}$ as $\mathcal{F} \otimes V_L^+$, $L = \mathbb{Z}\alpha$, $\langle \alpha, \alpha \rangle = 9$, with respect to involution $\alpha \to -\alpha$. Another description of this orbifold comes from a coset construction of $\mathfrak{so}(9)$, see Theorem 4. We obtain yet another realization from a certain $\mathcal{W}$-algebra obtained by Drinfeld-Sokolov reduction from $\mathfrak{osp}(1|8)$, see Theorem 5.

In the second part of the paper, we consider $\mathbb{Z}$-graded symplectic fermion vertex operator superalgebras. Recall the rank one symplectic fermion vertex superalgebra

$$SF = \Lambda(e(-1), e(-2), ..., f(-1), f(-2), ...)$$

generated by odd fields $e(z)$ and $f(z)$ subject to bracket relations

$$[e(i), f(j)]_+ = i\langle e, f\rangle \delta_{i+j,0}$$

where $\langle\ ,\ \rangle$ is skew-symmetric. Again we consider the invariant vertex superalgebra

$$(\mathcal{S}F \otimes \mathcal{S}F \otimes \mathcal{S}F)^{S_3}.$$

Our second main result is about the structure of this algebra. We prove in Theorem 6 that this orbifold is of type $(1^2, 2, 3^3, 4^3, 5^5, 6^4)$, meaning that we have a minimal strongly generated set of this type.

Throughout the paper, for brevity, we let $\mathcal{F}(n) := \underbrace{\mathcal{F} \otimes \cdots \otimes \mathcal{F}}_{n-times}$ and $\mathcal{SF}(n) = \underbrace{\mathcal{S}F \otimes \cdots \otimes \mathcal{S}F}_{n-times}$. We also use ϕ_i, $1 \leq i \leq n$, e_i, f_i to denote i-th component fermion inside the tensor product of ordinary and symplectic fermions, respectively.

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2 Warm-up: S_2-orbifold of $\mathcal{F}(2)$

This orbifold is well-known (see, for instance, [16]). If we set

$$\psi := \frac{1}{\sqrt{2}}(\phi_1 - \phi_2),$$

we see that the generator of S_2 acts via $\psi \mapsto -\psi$. As

$$\frac{1}{\sqrt{2}}(\phi_1 + \phi_2),$$

is fixed under the involution we immediately get

$$\mathcal{F}(2)^{S_2} = \mathcal{F}(1) \otimes \mathcal{F}(1)^{\mathbb{Z}_2} \cong \mathcal{F}(1) \otimes L_{Vir}\left(\frac{1}{2}, 0\right),$$

where $L_{Vir}(\frac{1}{2}, 0)$ is the simple Virasoro VOA of central charge $\frac{1}{2}$. In particular, this orbifold is of type $(\frac{1}{2}, 2)$. The standard proof in [16] uses unitarity and highest weight theory to conclude that the second tensor factor is generated by ω. As above argument cannot be used to other permutation groups, we offer another proof which uses classical invariant theory.

An application of the first fundamental theorem of invariant theory for $\mathcal{O}(1) \cong \mathbb{Z}_2 \cong S_2$ gives us an initial generating set of

$$\{\omega(a, b) | 0 \leq a < b\},$$

where

$$\omega(a, b) := \psi_{-\frac{1}{2}-a}\psi_{-\frac{1}{2}-b}\mathbb{1}.$$

The derivation operator allows us to reduce this generating set to

$$\{\omega(0, 2m+1) | m \geq 0\}.$$

Now if we set

$$\Omega(a, b) := Y(\omega(a, b), z),$$

we have the following relation

$$:\Omega(0, 2m+1)\Omega(0, 1): = -\frac{2m+5}{4m+6}\Omega(0, 2m+3) + \frac{2m+3}{2m+2}\Omega(1, 2m+2). \quad (1)$$

Now we replace

$$\Omega(1,2m+2) = \frac{\partial}{\partial z}\Omega(0,2m+2) - \Omega(0,2m+3)$$

which allows us to rewrite (1) as

$$\begin{aligned}\Omega(0,2m+3) = &\frac{2m+3}{(m+2)(6m+7)}((2m+3)\frac{\partial}{\partial z}\Omega(0,2m+2) \\ &- (2m+2)\,{}^{\circ}_{\circ}\Omega(0,2m+1)\Omega(0,1){}^{\circ}_{\circ}).\end{aligned} \quad (2)$$

Equivalently we have the following equation involving the states

$$\omega(0,2m+3) = \frac{1}{(m+2)(6m+7)}\left((2m+3)\omega(0,2m+2)_{-2}\mathbb{1} - \omega(0,2m+1)_{-1}\omega(0,1)\right). \quad (3)$$

From which it inductively follows that we only need the generator $\omega(0,1)$, as argued earlier.

3 Structure of the S_3-orbifold of $\mathcal{F}(3)$

We consider a tensor product of three free fermions $\mathcal{F}(3)$. As before we denote by ϕ_i the i-th component fermion so that $\mathcal{F}(3)$ is isomorphic to $\Lambda(\phi_1(-1/2),\phi_2(-1/2),\phi_3(-1/2),\ldots)$. The symmetric groups S_3 now acts via permuting the indices of ϕ_i, $1 \le i \le 3$. In fact, we can view $S_3 \subset \mathcal{O}(3)$, with orthogonal group $\mathcal{O}(3)$ acting in the usual way on $\mathrm{Span}\{\phi_1(-n-1/2),\phi_2(-n-1/2),\phi_3(-n-1/2)\}$.

Lemma 1 *The vertex algebra $\mathcal{F}(3)^{S_3}$ is generated by*

$$\begin{aligned}\omega_1(a) &:= \sum_{i=1}^{3}\phi_i(-a-1/2) \text{ for } a \ge 0\\ \omega_2(a,b) &:= \sum_{i=1}^{3}\phi_i(-a-1/2)\phi_i(-b-1/2) \text{ for } a > b \ge 0\\ \omega_3(a,b,c) &:= \sum_{i=1}^{3}\phi_i(-a-1/2)\phi_i(-b-1/2)\phi_i(-c-1/2) \text{ for } a > b > c \ge 0.\end{aligned} \quad (4)$$

Proof We begin with the following change of basis

$$\begin{aligned}\psi_0 &= \frac{1}{\sqrt{3}}(\phi_1+\phi_2+\phi_3)\\ \psi_1 &= \frac{1}{\sqrt{3}}(\phi_1+\eta^2\phi_2+\eta\phi_3)\\ \psi_2 &= \frac{1}{\sqrt{3}}(\phi_1+\eta\phi_2+\eta^2\phi_3),\end{aligned} \tag{5}$$

where η is a primitive third root of unity. In this new basis the vector ψ_0 is clearly fixed and generates a copy of the rank 1 free fermion algebra, $\mathcal{F}(1)$. Moreover the generators of $S_3 \cong D_3$ (viewed as Dihedral group acting on $\mathbb{R}^2$) act as follows

$$\begin{aligned}\tau_{23}\psi_1 = \psi_2, \qquad & \tau_{23}\psi_2 = \psi_1\\ \sigma_{123}\psi_1 = \eta\psi_1, \quad & \sigma_{123}\psi_2 = \eta^2\psi_2.\end{aligned} \tag{6}$$

It is clear that we have an initial decomposition of

$$\mathcal{F}(3)^{S_3} \cong \mathcal{F}(1) \otimes \mathcal{F}(2)^{D_3}. \tag{7}$$

As such, we will describe an initial set of generators for the orbifold $\mathcal{F}(2)^{D_3}$ and show that together with ψ_0 these may be used to construct (4). Further, these diagonalized generators will be used in our reduction calculations below.

The associated graded algebra of $\mathcal{F}(2)$ is the exterior algebra

$$\mathfrak{F}(2) = \bigwedge(x_1(m_1), x_2(m_2) | m_i \geq 0)$$

where we have the linear isomorphism

$$\pi : \mathcal{F}(2) \to \mathfrak{F}(2) \tag{8}$$

given by

$$\begin{aligned}\psi_1(-m_1-1/2)\cdots\psi_1(-m_k-1/2)\psi_2(-n_1-1/2)\cdots\psi_2(-n_\ell-1/2)\mathbb{1}\\ \mapsto x_1(m_1)\cdots x_1(m_k)x_2(n_1)\cdots x_2(n_\ell)\end{aligned}, \tag{9}$$

for $m_i, n_j \in \mathbb{Z}_{\geq 0}$. Now we recall (see [22] for the even case) that given a finite group acting on $\mathfrak{F}(2)$, the invariant subalgebra $\mathfrak{F}(2)^G$ is generated by the set of orbit sums of monomials. That is, by the elements

$$\mathbf{o}(m) = \sum_{g\in G} g \cdot m. \tag{10}$$

Given an arbitrary monomial $m = x_1(m_1)\cdots x_1(m_k)x_2(n_1)\cdots x_2(n_\ell) \in \mathfrak{F}(2)$, we see that

$$\mathbf{o}(m) =(1+\eta^{k-\ell}+\eta^{2(k-\ell)}) \\ (x_1(m_1)\cdots x_1(m_k)x_2(n_1)\cdots x_2(n_\ell)+x_2(m_1)\cdots x_2(m_k)x_1(n_1)\cdots x_1(n_\ell)), \tag{11}$$

which is nonzero if and only if $k-\ell \equiv 0 \pmod 3$. As such, we see that $\mathfrak{F}(2)^{D_3}$ is generated by

$$\begin{aligned} q_{k,3\ell}(\mathbf{r},\mathbf{s},\mathbf{t}) =&(x_1(r_1)x_2(s_1))\cdots(x_1(r_k)x_2(s_k))x_1(t_1)\cdots x_1(t_{3\ell}) \\ &+(x_2(r_1)x_1(s_1))\cdots(x_2(r_k)x_1(s_k))x_2(t_1)\cdots x_2(t_{3\ell}). \end{aligned} \tag{12}$$

If $\ell = 0$ then $q_{k,0}(\mathbf{r},\mathbf{s},\mathbf{0}) \in \mathfrak{F}(2)^{D_n}$ for all n and is thus in $\mathfrak{F}(2)^{\mathcal{O}(2)}$ (note that $\cup_{n\geq 2} D_n$ is dense in $\mathcal{O}(2)$).

An odd analogue to the first fundamental theorem of invariant theory for $\mathcal{O}(2)$, [25, 18], implies that these terms are in the subalgebra generated by the quadratic binomials

$$q_2(m,n) = x_1(m)x_2(n) + x_2(m)x_1(n). \tag{13}$$

So we have accounted for all of the generators from (12) of the form $q_{k,0}(\mathbf{r},\mathbf{s},\mathbf{0})$. Now we move onto those of the form $q_{k,3\ell}(\mathbf{r},\mathbf{s},\mathbf{t})$ for $\ell \neq 0$ the first of which is

$$q_3(m_1,m_2,m_3) = x_1(m_1)x_1(m_2)x_1(m_3) + x_2(m_1)x_2(m_2)x_2(m_3). \tag{14}$$

We will inductively show that all other invariants of the form (12) are in the subalgebra generated by the binomials $q_2(m_1,m_2)$ and $q_3(n_1,n_2,n_3)$ for $m_i, n_j \in \mathbb{Z}_{\geq 0}$. Suppose that we have an orbit sum $q_{k,3\ell}(\mathbf{r},\mathbf{s},\mathbf{t})$ with $k,\ell \neq 0$, which can be written as

$$(x_1(r_1)x_2(s_1))x_1(t_1)x_1(t_2)x_1(t_3)X_1 + (x_2(r_1)x_1(s_1))x_2(t_1)x_2(t_2)x_2(t_3)X_2, \tag{15}$$

where X_i are monomials appropriate to complete the term. Now denote

$$q_3^X(a,b,c) = x_1(a)x_1(b)x_1(c)X_1 + x_2(a)x_2(b)x_2(c)X_2, \tag{16}$$

where is invariant of degree two less than our orbit sum. Now observe that

$$\begin{aligned} q_{k,3\ell}(\mathbf{r},\mathbf{s},\mathbf{t}) =&\frac{1}{2}q_2(r_1,s_1)q_3^X(t_1,t_2,t_3) + \frac{1}{2}q_2(r_1,t_3)q_3^X(s_1,t_1,t_2) \\ &+ \frac{1}{2}q_2(s_1,t_3)q_3^X(r_1,t_2,t_3). \end{aligned} \tag{17}$$

Now suppose that we have an orbit sum $q_{k,3\ell}(\mathbf{r},\mathbf{s},\mathbf{t})$ with $\ell \geq 2$, which can be written (up to a sign) as

$$x_1(t_1)\cdots x_1(t_6)X_1 + x_2(t_1)\cdots x_2(t_6)X_2, \tag{18}$$

where X_i are monomials appropriate to complete the term. Now denote

$$
\begin{aligned}
q_2^X(a,b) &= x_1(a)x_2(b)X_1 + x_2(a)x_1(b)X_2 \\
q_3^X(a,b,c) &= x_1(a)x_1(b)x_1(c)X_1 + x_2(a)x_2(b)x_2(c)X_2
\end{aligned} \tag{19}
$$

which are invariants of degree 3 and 2 less than our orbit sum, respectively. Now observe that

$$
\begin{aligned}
q_{k,3\ell}(\mathbf{r},\mathbf{s},\mathbf{t}) = {} & \frac{1}{2}q_2(t_1,t_2)q_2(t_3,t_6)q_2^X(t_4,t_5) - \frac{1}{2}q_2(t_1,t_4)q_2(t_2,t_6)q)2^X(t_3,t_5) \\
& - \frac{1}{2}q_2(t_1,t_6)q_2(t_2,t_3)q_2^X(t_4,t_5) - q_2(t_1,t_6)q_2(t_2,t_4)q_2^X(t_3,t_5) \\
& - q_2(t_1,t_6)q_2(t_2,t_5)q_2^X(t_3,t_4) - \frac{1}{4}q_2(t_2,t_4)q_2(t_3,t_6)q_2^X(t_1,t_5) \\
& - \frac{3}{4}q_2(t_3,t_6)q_2(t_2,t_5)q_2^X(t_1,t_4) + \frac{1}{4}q_2(t_4,t_5)q_2(t_2,t_3)q_2^X(t_1,t_6) \\
& - \frac{1}{4}q_2(t_4,t_5)q_2(t_2,t_3)q_2^X(t_1,t_6) - \frac{1}{4}q_2(t_4,t_5),q_2(t_2,t_6)q_2^X(t_1,t_3) \\
& - \frac{3}{4}q_2(t_4,t_5)q_2(t_3,t_6)q_2^X(t_1,t_2) + \frac{1}{4}q_2(t_4,t_6)q_2(t_3,t_5)q_2^X(t_1,t_2) \\
& + \frac{1}{2}q_2(t_5,t_6)q_2(t_2,t_4)q_2^X(t_1,t_3) + \frac{3}{4}q_2(t_5,t_6)q_2(t_3,t_4)q_2^X(t_1,t_2) \\
& - q_3(t_1,t_2,t_4)q_3^X(t_3,t_5,t_6).
\end{aligned} \tag{20}
$$

Now, given an arbitrary orbit sum $q_{k,3\ell}(\mathbf{r},\mathbf{s},\mathbf{t})$, we can inductively use (17) to reduce it to an algebraic combination of $q_2(m_1,m_2)$ and orbit sums $q_{0,3\ell}(\mathbf{r}^{'},\mathbf{s}^{'},\mathbf{t}^{'})$ and then (20) can be used to write $q_{0,3\ell}(\mathbf{r}^{'},\mathbf{s}^{'},\mathbf{t}^{'})$ in terms of $q_2(m_1,m_2)$ and $q_3(n_1,n_2,n_3)$ implying that these are the only required generators.

The preimages of these terms in $\mathcal{F}(2)$, along with the linear generator, are

$$
\begin{aligned}
\omega_1^0(a) &= \psi_0(-a-1/2) \\
\omega_2^0(a,b) &= \psi_1(-a-1/2)\psi_2(-b-1/2)\mathbb{1} + \psi_2(-a-1/2)\psi_1(-b-1/2)\mathbb{1} \\
\omega_3^0(a,b,c) &= \sum_{i=1}^{2}\psi_i(-a-1/2)\psi_i(-b-1/2)\psi_i(-c-1/2)\mathbb{1}.
\end{aligned} \tag{21}
$$

Now observe that we have the following translation between (4) and (21)

$$
\begin{aligned}
\omega_1(a) = {} & \sqrt{3}\omega_1^0(a) \\
\omega_2(a,b) = {} & \omega_2(a,b) + \omega_1^0(a)_{-1}\omega_1^0(b) \\
\omega_3(a,b,c) = {} & \frac{1}{\sqrt{3}}(\omega_3^0(a,b,c) + \omega_1^0(a)_{-1}\omega_2^0(b,c) - \omega_1^0(b)\omega_2^0(a,c) \\
& + \omega_1^0(c)\omega_2^0(a,b) + \omega_1^0(a)_{-1}\omega_1^0(b)_{-1}\omega_1^0(c)).
\end{aligned} \tag{22}
$$

We now work to minimize this generating set by way of quantum corrections to the odd analogues of the classical relations found in [21]. The results are summarized in the following Lemma.

Lemma 2 *The generators described in (4) may be replaced with the set*

$$\omega_1(0), \omega_2(0,1), \omega_2(0,3), \omega_3(0,1,2). \tag{23}$$

Proof For $a \geq 0$ the reduction of the linear generators to the single vector $\omega_1(0)$ is trivial by way of the translation operator. Moving on to the quadratic generators, using methods from [8] we can initial reduce these to the set $\omega_2(0, 2m+1)$ for $m \geq 0$.

To reduce the quadratic generators further down to $\omega_2(0,1), \omega_2(0,3)$ we can simply use a result of Linshaw [18]. Here we give a direct proof for completeness.

Further, if we set $\Omega_2(a,b) = Y(\omega_2^0(a,b), z)$ we have

$$\begin{aligned} \Omega_2(0,5) = &-\frac{2}{15} {}^{\circ}_{\circ}\Omega_2(0,1)\Omega_2(0,1)\Omega_2(0,1){}^{\circ}_{\circ} - \frac{14}{15} {}^{\circ}_{\circ}\Omega_2(0,1)\Omega_2(0,3){}^{\circ}_{\circ} \\ &+ \frac{3}{5} {}^{\circ}_{\circ}\partial^2\Omega_2(0,1)\Omega_2(0,1){}^{\circ}_{\circ} + \frac{37}{15}\partial^2\Omega_2(0,3) - \frac{53}{30}\partial^4\Omega_2(0,1). \end{aligned} \tag{24}$$

Also, we have for all $m \geq 2$

$$\Omega_2(0, m+1) = \frac{1}{2m^2+4m}\left(3\Omega_2(0,1)_0 - \Omega_2(0,3)_1\right)\Omega_2(0,m), \tag{25}$$

allowing us to lift the relation (24) to a higher weight and establish the quadratic portion of (23). In the process we use the fact that the modes of quadratic operators form a Lie algebra.

For cubic operators we need a different approach as cubic operators do not close a Lie algebra. We first consider cubic generators beginning with the observation that all of the cubic vectors of weight at most $\frac{13}{2}$ can be written as a linear combination of $\omega_3(0,1,2)$ and $\omega_3(0,1,4)$ with the translation operator applied as necessary and more generally we may only consider vectors $\omega_3(0,a,b)$ with $0 < a < b$. Now we set $\Omega_3(a,b,c) = Y(\omega_3^0(a,b,c), z)$.

Our main tool for reducing the cubic generating set will be the relation among the generators of the associated graded algebra given by

$$\begin{aligned} D_5^C(a_1,a_2,a_3,a_4,a_5) := &\, q_2(a_1,a_2)q_3(a_3,a_4,a_5) + q_2(a_1,a_5)q_3(a_2,a_3,a_4) \\ &- q_2(a_2,a_5)q_3(a_1,a_3,a_4) - q_2(a_3,a_4)q_3(a_1,a_2,a_5) \\ &- q_2(a_3,a_5)q_3(a_1,a_2,a_4) + q_2(a_4,a_5)q_3(a_1,a_2,\alpha_3) \\ &= 0, \end{aligned} \tag{26}$$

which when lifted to fields corresponding to elements in the orbifold yields the expression

$$\begin{aligned} D_5(a_1,a_2,a_3,a_4,a_5) := {}& {}_\circ^\circ \Omega_2(a_1,a_2)\Omega_3(a_3,a_4,a_5){}_\circ^\circ + {}_\circ^\circ \Omega_2(a_1,a_5), \Omega_3(a_2,a_3,a_4){}_\circ^\circ \\ & - {}_\circ^\circ \Omega_2(a_2,a_5)\Omega_3(a_1,a_3,a_4){}_\circ^\circ - {}_\circ^\circ \Omega_2(a_3,a_4)\Omega_3(a_1,a_2,a_5){}_\circ^\circ \\ & - {}_\circ^\circ \Omega_2(a_3,a_5)\Omega_3(a_1,a_2,a_4){}_\circ^\circ + {}_\circ^\circ \Omega_2(a_4,a_5)\Omega_3(a_1,a_2,\alpha_3){}_\circ^\circ, \end{aligned} \tag{27}$$

that because of (26) we may write

$$D_5(a_1,a_2,a_3,a_4,a_5) = \sum_{\substack{b_1,b_2,b_3 \geq 0 \\ b_1+b_2+b_2 = a_1+\cdots+a_5+2}} \lambda_{b_1,b_2,b_3} \Omega_3(b_1,b_2,b_3). \tag{28}$$

It will also be helpful to order the fields $\Omega_3(a,b,c)$ lexicographically by their entries.

Before embarking on our argument involving the expressions $D_5(a_1,a_2,a_3,a_4,a_5)$, we make an initial observation that for $0 \leq a < b < c$ we may write

$$\Omega_3(a,b,c) = \partial\Omega_3(a-1,b,c) - \Omega_3(a-1,b+1,c) - \Omega_3(a-1,b,c+1) \tag{29}$$

which applied iteratively, allows us to initially reduce our generating set to fields of the form $\Omega_3(0,b,c)$ for $1 < b < c$.

We may use $D_5(1,0,2,0,1)$ to form the lowest weight decoupling relation for the initial generating set (4)

$$\Omega_3(0,1,4) = -\frac{12}{17} {}_\circ^\circ \Omega_2(0,1)\Omega_3(0,1,2){}_\circ^\circ + \frac{10}{17}\partial^2\Omega_3(0,1,2). \tag{30}$$

Expanding the expression $D_5(1,0,a,0,b)$, for $1 \leq a \leq b+2$ gives

$$\begin{aligned} D_5(1,0,a,0,b) = {}& \left(-\frac{(-1)^a}{a+2} - \frac{1}{a+2} - \frac{(-1)^a}{a+1} - \frac{1}{a+1}\right)\Omega_3(0,a+2,b) \\ & - \left(\frac{(-1)^a}{a+1} + \frac{(-1)^b}{b+1} + \frac{(-1)^b}{b+1} - 4\right)\Omega_3(0,a+1,b+1) + \cdots \end{aligned} \tag{31}$$

where the missing terms have second entry less than $a+1$. In the case that a is even this expression can be used to solve for $\Omega_3(0,a+2,b)$ in terms of fields lower in the order. If a is odd, the coefficient of $\Omega(0,a+2,b)$ in (31) is zero and thus we may use this equation to write $\Omega_3(0,a+1,b+1)$ in terms of fields lower in the ordering. All that remains is to show that for $a \geq 5$ $\Omega_3(0,1,a)$ may be eliminated from the generating set. We use a similar argument to the one found in [21], considering the expressions $D_5(1,0,a-2,0,1)$, $D_5(1,0,a-3,0,2)$, $D_5(2,0,a-3,0,1)$, $D_5(2,0,a-4,0,2)$, and $D_5(1,0,a-4,0,2)$ all of which may be written as linear combinations of $\Omega_3(0,k,a+1-k)$ for $1 \leq k \leq 5$ and derivatives of fields of lower weight. We form the matrix A whose (i,j) entry is the coefficient of $\Omega_3(0,i,a+1-i)$ from the j^{th} expression listed above. We have

$$\det\ A = -\frac{(a+2)\left(1065a^5+3550a^4-47884a^3+108085a^2-77316a+9180\right)}{40(a-3)(a-2)^2(a-1)^2a} \tag{32}$$

if a is even and

$$\det\ A = -\frac{1065a^5+8620a^4-66306a^3+132611a^2-112314a+42120}{40(a-3)(a-2)(a-1)^2a} \tag{33}$$

if a is odd. In each of these cases, this allows us to write $\Omega_3(0,k,a+1-k)$ for $1 \leq k \leq 5 \leq a$ in terms of fields of lower weight. This along with (31) allows us to inductively eliminate all cubic generators not described in (23). □

This leads us to the following result.

Theorem 1 *The orbifold $\mathcal{F}(3)^{S_3}$ is strongly generated by a minimal generating set of vectors of weight $\frac{1}{2}, 2, 4$ and $\frac{9}{2}$. As a vertex algebra it is isomorphic to the tensor product of $\mathcal{F}(1)$ and a $\mathcal{W}$-(super)algebra of type $(2, 4, \frac{9}{2})$.*

In a future work, we will examine the algebra W described in the Theorem 1, which is of central charge 1 and minimally strongly generated by the primary vectors

$$\begin{aligned}
\omega &= -\frac{1}{2}\omega_2^0(0,1) \\
j &= \omega_2^0(0,3) - 2\omega_2^0(0,1)_{-3}\mathbb{1} + \frac{7}{3}\omega_2^0(0,1)_{-1}\omega_2^0(0,1) \\
c &= \omega_3^0(0,1,2).
\end{aligned} \tag{34}$$

We may also calculate the character of $\mathcal{F}(3)^{S_3}$.

Theorem 2 *We have*

$$\chi_{\mathcal{F}(3)^{S_3}}(q) = \frac{q^{-\frac{1}{16}}}{6}\left(\prod_{n\geq 1}(1+q^{n-\frac{1}{2}})^3 + 3\prod_{n\geq 1}(1-q^{2n-1})(1+q^{n-\frac{1}{2}}) + 2\prod_{n\geq 1}(1+q^{3n-\frac{3}{2}})\right). \tag{35}$$

Proof Proof is analogous to the proof of the character formula of $\mathcal{H}(3)^{S_3}(q)$ in [21] so we omit it here. □

Using methods similar to those above it follows that the orbifold $\mathcal{F}(3)^{\mathbb{Z}_3}$ has a minimal strong generating set given by

$$\begin{aligned}\widehat{\omega_1}(0) &= \psi_0\left(-\frac{1}{2}\right)\\ \widehat{\omega_2}(0,0) &= \psi_1\left(-\frac{1}{2}\right)\psi_2\left(-\frac{1}{2}\right)\\ \widehat{\omega_3^1}(0,1,2) &= \psi_1\left(-\frac{5}{2}\right)\psi_1\left(-\frac{3}{2}\right)\psi_1\left(-\frac{1}{2}\right)\\ \widehat{\omega_3^2}(0,1,2) &= \psi_2\left(-\frac{5}{2}\right)\psi_2\left(-\frac{3}{2}\right)\psi_2\left(-\frac{1}{2}\right)\end{aligned} \tag{36}$$

and thus $\mathcal{F}(3)^{\mathbb{Z}_3} \cong \mathcal{F}(1) \otimes W$ where W is of type 1, $\frac{9}{2}$, $\frac{9}{2}$ and

$$\chi_{\mathcal{F}(3)^{\mathbb{Z}_3}}(q) = \frac{q^{-\frac{1}{16}}}{3}\left(\prod_{n\geq 1}(1+q^{n-\frac{1}{2}})^3 + 2\prod_{n\geq 1}(1+q^{3(n-\frac{1}{2})})\right) \tag{37}$$

3.1 Bosonic description of $\mathcal{F}(3)^{S_3}$

The $\mathbb{Z}_3$-orbifold of $\mathcal{F}(3)$ can be now used to give another description of $\mathcal{F}(3)^{D_3} = \mathcal{F} \otimes \mathcal{F}(2)^{D_3}$.

Theorem 3 *We have*

$$\mathcal{F}(3)^{D_3} \cong \mathcal{F} \otimes V_{3\mathbb{Z}}^{+}$$

where $V_{3\mathbb{Z}}^{+}$ is the fixed point subalgebra of the lattice vertex superalgebra $V_{3\mathbb{Z}}$ under the involution $\alpha \to -\alpha$, where α is generator of $3\mathbb{Z}$.

Proof We already established that two generating vectors of weight $\frac{9}{2}$ are primary. It is also easy to see that they are of charge ± 3 highest weight vectors for the generator $h := \psi_1\left(-\frac{1}{2}\right)\psi_2\left(-\frac{1}{2}\right)$. Denote by $M(1)$ the Heisenberg subalgebra generated by h. We have conformal embedding $M(1) \subset \mathcal{F}(2)^{\mathbb{Z}_3}$ which decomposes as a direct sum of irreducible $M(1)$-modules

$$\mathcal{F}(2)^{\mathbb{Z}_3} \cong \oplus_{\lambda\in S} M(1,\lambda),$$

where $\lambda \in \mathbb{Z}$ and $\pm 3 \in S$ for some set S. Since $\mathcal{F}(2)^{\mathbb{Z}_3}$ is generated by h and two highest weight vectors of charge ± 3 and the charge is additive we conclude that $\mathcal{F}(2)^{\mathbb{Z}_3}$ is contained inside $\oplus_{m\in\mathbb{Z}} M(1,3m) \cong V_{3\mathbb{Z}}$, which itself has $\frac{1}{2}\mathbb{Z}$-graded vertex superalgebra structure. The rest follows from the simplicity of $\mathcal{F}(2)^{\mathbb{Z}_3}$ and a uniqueness property of lattice vertex algebras as in [10, Section 5], that is

$$\mathcal{F}(2)^{\mathbb{Z}_3} \cong V_{3\mathbb{Z}}$$

as vertex algebras. Finally, we observe that there is an automorphism in D_3 which acts as $h \to -h$, so we have the claim. □

Remark 1 Clearly we can give another proof for strong generation for $\mathcal{F}(2)^{D_3}$ using lattice vertex algebra structure.

Comparing two characters of the bosonic and fermionic side for the $\mathbb{Z}_3$-orbifold gives the following q-series identity.

Corollary 1 (Boson-Fermion correspondence for characters)

$$\frac{1}{3}\prod_{n\geq 1}(1+q^{n-\frac{1}{2}})^2+\frac{2}{3}\prod_{n\geq 1}\frac{(1+q^{3(n-\frac{1}{2})})}{(1+q^{n-\frac{1}{2}})}=\frac{\sum_{n\in\mathbb{Z}}q^{\frac{9n^2}{2}}}{(q;q)_\infty}.$$

3.2 $\mathcal{F}(3)^{S_3}$ as a coset vertex superalgebra

Here we give another description of the orbifold algebra using results of Adamovic and Perse [4]. Similar result was also predicted by physicists [24].

Theorem 4 *As vertex superalgebras,*

$$\mathcal{F}(2)^{D_3}\cong S\mathrm{Com}(L_{so(9)_2},L_{so(9)_1}\otimes L_{so(9)_1}),$$

where $S\mathrm{Com}(L_{so(9)_2},L_{so(9)_1}\otimes L_{so(9)_1})$ *denotes a simple current extension of the vertex operator algebra* $\mathrm{Com}(L_{so(9)_2},L_{so(9)_1}\otimes L_{so(9)_1})$ *and* $L_{so(9)_k}$ *is the simple affine Lie algebra of level* k *and type* $B_4^{(1)}$.

Proof This follows directly from [4, Section 4], specifically Corollary 2:

$$\mathrm{Com}(L_{so(9)_2},L_{so(9)_1}\otimes L_{so(9)_1})\cong V_{6\mathbb{Z}}^+,$$

together with Theorem 3, and decomposition

$$V_{3\mathbb{Z}}=V_{6\mathbb{Z}}\oplus V_{6\mathbb{Z}+3}.$$

3.3 $\mathcal{F}(3)^{S_3}$ via Drinfeld-Sokolov reduction

In the physics literature it is often quoted that the Drinfeld-Sokolov reduction of $\widehat{\mathfrak{osp}(1|2n)}$ Lie superalgebra at level k is a $\mathcal{W}$-superalgebra of type $(2,4,\ldots,2n,\frac{n+1}{2})$ (cf. [6]) and that it coincide with Fateev-Lukyanov algebra defined via Miura transformation. This $\mathcal{W}$-algebra exists generically and it can be described as the kernel of screenings - for a rigorous proof see [13].

We are interested in the $n=4$ case. Following [14] and [13] it can be easily shown that the central charge of $\mathcal{W}^k(osp(1|2k),f_{reg})$, where k is the level and f_{reg} is a regular nilpotent element [13], is given by

$$c = \frac{-9(55+14k)(65+16k)}{4(9+2k)}.$$

Solving for $c = 1$ gives $k = -\frac{63}{16}$ as one of the solutions (the other solution is $k = -\frac{73}{18}$). For this value of k this vertex algebra is not simple. More precisely,

Theorem 5

$$\mathcal{F}(2)^{D_3} \cong \mathcal{W}_{-\frac{63}{16}}(\mathfrak{osp}(1|8), f_{reg}),$$

where $W_{-\frac{63}{16}}(\mathfrak{osp}(1|8), f_{reg})$, *is the simple quotient of the universal* $\mathcal{W}$*-algebra* $\mathcal{W}^{-\frac{63}{16}}(\mathfrak{osp}(1|8), f_{reg})$.

Proof The proof is straightforward computation. By Theorem 6.4 of [13], the universal $\mathcal{W}$-algebra $\mathcal{W}^{-\frac{63}{16}}(\mathfrak{osp}(1|8), f)$ is strongly generated by fields of weight $\frac{9}{2}$, 2, 4, 6, and 8 which are denoted by G, W_6, W_4, W_2, and W_0 respectively. The field $\frac{1}{2}W_6$ is the conformal element and we will denote it by L. Further, we relabel the weight 4 generator $W_4 = W$. Furthermore a computer calculation shows that

$$\begin{aligned}\widetilde{W}_2 =& W_2 - \frac{7}{120}\partial^2 W + \frac{3017}{77760}\partial^4 L - \frac{31}{90}{:}LW{:} + \frac{5159}{6480}{:}\partial^2 LL{:}\\ &- \frac{497}{810}{:}\partial L\partial L{:} - \frac{49}{180}{:}LLL{:}\end{aligned} \tag{38}$$

and

$$\begin{aligned}\widetilde{W}_0 =& W_0 - \frac{154105}{639824}\partial^2 W_2 + \frac{59346175}{2487635712}\partial^4 W - \frac{1120613725}{537329313792}\partial^6 L\\ &- \frac{877345}{1439604}{:}LW_2{:} + \frac{964075}{8637624}{:}WW{:} + \frac{170975}{12956436}{:}LLW{:}\\ &- \frac{12284125}{34550496}{:}(\partial^2 L)W{:} - \frac{33422375}{310954464}{:}L(\partial^2 W){:} - \frac{10892875}{103651488}{:}LLLL{:}\\ &+ \frac{200351725}{466431696}{:}(\partial^2 L)LL{:} - \frac{141495725}{466431696}{:}(\partial L)(\partial L)L{:} - \frac{571813025}{22388721408}{:}(\partial^4 L)L{:}\\ &+ \frac{83651225}{1865726784}{:}(\partial^3 L)(\partial L){:} - \frac{366545725}{4975271424}{:}(\partial^2 L)(\partial^2 L){:}\end{aligned} \tag{39}$$

are singular and thus generate a proper VOA ideal inside $\mathcal{W}^{-\frac{63}{16}}(\mathfrak{osp}(1|8), f)$.

The rest is comparing OPEs among $2, 4$ and $\frac{9}{2}$ generators. Since they agree, modulo the ideal, we have a surjective map from the universal vertex algebra $W^{-\frac{63}{16}}(\mathfrak{osp}(1|8), f)$ to $\mathcal{F}(2)^{D_3}$, sending generators of weight 2, 4 and 9/2 to the corresponding generators of the orbifold and, in light of (38) and (39), $\widetilde{W}_2$ and $\widetilde{W}_0$ are mapped to zero. This map is well-defined. The rest follows from the simplicity of the orbifold algebra. □

4 Symmetric Orbifolds of Symplectic Fermions

The rank 3 symplectic fermion, $\mathcal{SF}(3)$, vertex operator algebra is generated by the odd vectors $e_i(-1)$ and $f_i(-1)$ for $1 \leq i \leq 3$ with vertex operators

$$\begin{aligned} Y(e_i(-1), z) &= \sum_{n\in\mathbb{Z}} e_i(n) z^{-n-1} \\ Y(f_i(-1), z) &= \sum_{n\in\mathbb{Z}} f_i(n) z^{-n-1}, \end{aligned} \tag{40}$$

subject to (anti-)commutation relations

$$\begin{aligned} [e_i(m), e_j(n)]_+ &= [f_i(m), f_j(n)]_+ = 0 \\ [e_i(m), f_j(n)]_+ &= m\delta_{i,j}\delta_{m+n,0}. \end{aligned} \tag{41}$$

Some invariant theory argument implies that we have the following initial set of strong generators for the orbifold $\mathcal{SF}(3)$ may be taken to be

$$\begin{aligned} \omega_e^1(a) &= \sum_{i=1}^{3} e_i(-a-1) \\ \omega_f^1(a) &= \sum_{i=1}^{3} f_i(-a-1) \\ \omega_{e,e}^2(a,b) &= \sum_{i=1}^{3} e_i(-a-1)e_i(-b-1) \\ \omega_{e,f}^2(a,b) &= \sum_{i=1}^{3} e_i(-a-1)f_i(-b-1) \\ \omega_{f,f}^2(a,b) &= \sum_{i=1}^{3} f_i(-a-1)f_i(-b-1) \\ \omega_{e,e,e}^3(a,b,c) &= \sum_{i=1}^{3} e_i(-a-1)e_i(-b-1)e_i(-c-1) \\ \omega_{e,e,f}^3(a,b,c) &= \sum_{i=1}^{3} e_i(-a-1)e_i(-b-1)f_i(-c-1) \\ \omega_{e,f,f}^3(a,b,c) &= \sum_{i=1}^{3} e_i(-a-1)f_i(-b-1)f_i(-c-1) \\ \omega_{f,f,f}^3(a,b,c) &= \sum_{i=1}^{3} f_i(-a-1)f_i(-b-1)f_i(-c-1) \end{aligned} \tag{42}$$

for $a, b, c \geq 0$. Observe that the conformal vector is given by $\omega = \frac{1}{2}\omega_{1,2,2}(0,0)$. The following change of basis of the generating set will allow for an efficient reduction in the initial strong generating set (42).

$$\begin{aligned} E_0(-1) &= \frac{1}{\sqrt{3}}(e_1(-1) + e_2(-1) + e_3(-1)) \\ E_1(-1) &= \frac{1}{\sqrt{3}}(e_1(-1) + \eta^2 e_2(-1) + \eta e_3(-1)) \\ E_2(-1) &= \frac{1}{\sqrt{3}}(e_1(-1) + \eta e_2(-1) + \eta^2 e_3(-1)). \end{aligned} \tag{43}$$

and

$$\begin{aligned} F_0(-1) &= \frac{1}{\sqrt{3}}(f_1(-1) + f_2(-1) + f_3(-1)) \\ F_1(-1) &= \frac{1}{\sqrt{3}}(f_1(-1) + \eta^2 f_2(-1) + \eta f_3(-1)) \\ F_2(-1) &= \frac{1}{\sqrt{3}}(f_1(-1) + \eta f_2(-1) + \eta^2 f_3(-1)). \end{aligned} \tag{44}$$

where η is a primitive third root of unity. Using this generating set we have $\sigma \cdot E_0(-1) = E_0(-1)$ and $\sigma \cdot F_0(-1) = F_0(-1)$ for all $\sigma \in S_3$. Examining the action of the generators of S_3 on these new generators of $\mathcal{SF}(3)$, we have

$$\begin{aligned} \tau_{12}E_1(-1) &= E_2(-1) \\ \tau_{12}F_1 &= F_2(-1). \end{aligned} \tag{45}$$

and

$$\begin{aligned} \sigma_{123}E_1(-1) &= \eta E_1(-1) \\ \sigma_{123}E_2(-1) &= \eta^2 E_2(-1) \\ \sigma_{123}F_1(-1) &= \eta F_1(-1) \\ \sigma_{123}F_2(-1) &= \eta^2 F_2(-1). \end{aligned} \tag{46}$$

From this action, we see that an initial generating set for the orbifold may alternatively taken to be

$$
\begin{aligned}
\Omega^1_E(a) &= E_0(-a-1)\\
\Omega^1_F(a) &= F_0(-a-1)\\
\Omega^2_{E,E}(a,b) &= E_1(-a-1)E_2(-b-1)+E_2(-a-1)E_1(-b-1)\\
\Omega^2_{E,F}(a,b) &= E_1(-a-1)F_2(-b-1)+E_2(-a-1)F_1(-b-1)\\
\Omega^2_{F,F}(a,b) &= F_1(-a-1)F_2(-b-1)+F_2(-a-1)F_1(-b-1)\\
\Omega^3_{E,E,E}(a,b,c) &= \sum_{i=1}^{2} E_i(-a-1)E_i(-b-1)E_i(-c-1)\\
\Omega^3_{E,E,F}(a,b,c) &= \sum_{i=1}^{2} E_i(-a-1)E_i(-b-1)F_i(-c-1)\\
\Omega^3_{E,F,F}(a,b,c) &= \sum_{i=1}^{2} E_i(-a-1)F_i(-b-1)F_i(-c-1)\\
\Omega^3_{F,F,F}(a,b,c) &= \sum_{i=1}^{2} F_i(-a-1)F_i(-b-1)F_i(-c-1)
\end{aligned}
\tag{47}
$$

for $a,b,c \geq 0$

Of course the linear portion of the generating set can be immediately reduced to $\Omega_{1,1}(0)$ and $\Omega_{2,1}(0)$. The following lemmas describe the reduction of the quadratic and cubic terms in the generating set.

Lemma 3 *The quadratic elements in the generating set (47) may be replaced with the homogeneous quadratic generators*

$$\Omega^2_{E,E}(0,1), \Omega^2_{E,E}(0,3), \Omega^2_{F,F}(0,1), \Omega^2_{F,F}(0,3) \tag{48}$$

and the heterogeneous quadratic generators

$$\Omega^2_{E,F}(0,0), \Omega^2_{E,F}(0,1), \Omega^2_{E,F}(0,2), \Omega^2_{E,F}(0,3). \tag{49}$$

Proof We begin by reducing the set of homogeneous generators $\Omega^2_{E,E}(a,b)$ and $\Omega^2_{F,F}(a,b)$ for $a,b \geq 0$ by focusing on the generators $\Omega^2_{E,E}(a,b)$, as the others will follow similarly. We have the following decoupling relation

$$
\begin{aligned}
\Omega^2_{E,E}(0,5) = &-\frac{1}{140}\Omega^2_{E,E}(0,1)_{-1}\Omega^2_{E,F}(0,0)_{-1}\Omega^2_{E,F}(0,0) - \frac{1}{140}\Omega^2_{E,E}(0,1)_{-1}\Omega^2_{E,F}(0,2)\\
&-\frac{1}{140}\Omega^2_{E,E}(0,1)_{-1}\Omega^2_{E,F}(2,0) - \frac{1}{14}\Omega^2_{E,E}(0,3)_{-1}\Omega^2_{E,F}(0,0).
\end{aligned}
$$

Moving to the heterogeneous quadratic generators, we begin we the following

$$\begin{aligned}\Omega^2_{E,F}(0,4) = &-\frac{1}{36}\Omega^2_{E,F}(0,0)_{-1}\Omega^2_{E,F}(0,0)_{-1}\Omega^2_{E,F}(0,0) - \frac{1}{12}\Omega^2_{E,F}(0,0)_{-1}\Omega^2_{E,F}(0,2)\\ &- \frac{1}{12}\Omega^2_{E,F}(0,0)_{-1}\Omega^2_{E,F}(2,0) + \frac{1}{2}\Omega^2_{E,F}(0,3)_{-2}\mathbb{1} - \frac{5}{18}\Omega^2_{E,F}(0,2)_{-3}\mathbb{1}\\ &+ \frac{1}{6}\Omega^2_{E,F}(0,1)_{-4}\mathbb{1} - \frac{1}{6}\Omega^2_{E,F}(0,0)_{-5}\mathbb{1}.\end{aligned}$$

Finally operators similar to those used in the proof of Lemma 2 can be used to construct higher weight decoupling relations and finish the argument.

Lemma 4 *The cubic elements in the generating set (47) may be replaced with the homogeneous cubic generators*

$$\Omega^3_{E,E,E}(0,1,2), \Omega^3_{F,F,F}(0,1,2). \tag{50}$$

and heterogeneous cubic generators

$$\Omega^3_{E,E,F}(0,1,0), \Omega^3_{E,E,F}(0,2,0), \Omega^3_{E,E,F}(1,2,0) \tag{51}$$

and

$$\Omega^3_{E,F,F}(0,1,0), \Omega^3_{E,F,F}(0,2,0), \Omega^3_{E,F,F}(0,1,2). \tag{52}$$

Proof The reduction of homogeneous cubic generators follows similarly to the $\mathcal{F}(3)$ case as described above. So we move on to the heterogeneous cubic generators. Observe that all weight 6 vectors among the generators $\Omega^3_{E,E,F}(a,b,c)$ are in the list

$$\Omega^3_{E,E,F}(0,3,0), \Omega^3_{E,E,F}(1,2,0), \Omega^3_{E,E,F}(0,2,1), \Omega^3_{E,E,F}(0,1,2). \tag{53}$$

Taking linear combinations of $\Omega^3_{E,E,F}(0,1,0)_{-3}\mathbb{1}$ and $\Omega^3_{E,E,F}(0,2,0)_{-2}\mathbb{1}$ will allow us to eliminate any two of these from our generating set, further the equation

$$\begin{aligned}\Omega^3_{E,E,F}(0,3,0) = &-\frac{1}{5}\Omega^2_{E,F}(0,0)_{-1}\Omega^3_{E,E,F}(0,1,0) - \frac{1}{20}\Omega^3_{E,E,F}(0,1,0)_{-3}\mathbb{1}\\ &- \frac{1}{20}\Omega^3_{E,E,F}(0,2,0)_{-2}\mathbb{1}\end{aligned} \tag{54}$$

allows us to eliminate $\Omega^3_{E,E,F}(0,3,0)$. We keep $\Omega^3_{E,E,F}(1,2,0)$ as our weight 6 cubic generator.

Now observe that all weight 7 vectors among the generators $\Omega^3_{E,E,F}(a,b,c)$ are in the list

$$\Omega^3_{E,E,F}(0,4,0), \Omega^3_{E,E,F}(1,3,0), \Omega^3_{E,E,F}(0,3,1), \Omega^3_{E,E,F}(0,1,3), \Omega^3_{E,E,F}(0,2,2). \tag{55}$$

Taking linear combinations of the translation operator applied to vectors on the list (53) allows us to eliminate all but one of these. Further the equation

$$\Omega^2_{E,F}(0,0)_{-1}\Omega^3_{E,E,F}(0,2,0)+\Omega^3_{E,E,F}(0,2,2)+3\Omega^3_{E,E,F}(0,4,0)=0 \quad (56)$$

allows us to eliminate the final weight 7 vector. Finally an argument following the outline of Lemma 2 using relations analogous to $D_5(a_1,a_2,a_3,a_4,a_5)$ allows us to eliminate all higher weight generators.

Theorem 6 *The vertex operator algebra* $\mathcal{SF}(3)^{S_3}$ *is simple of type* $(1^2,2,3^3,4^3,5^5,6^4)$. *It is isomorphic to* $\mathcal{SF}(1)\otimes W$ *where* W *is of type* $(2,3^3,4^3,5^5,6^4)$.

Again we can easily compute the orbifold character as in [21].

Theorem 7 *We have*

$$\chi_{\mathcal{SF}(3)^{S_3}}(q)=\frac{q^{1/4}}{6}\left(\prod_{n\geq 1}(1+q^n)^6+3\prod_{n\geq 1}(1+q^n)^2(1-q^{2n})^2+2\prod_{n\geq 1}(1+q^{3n})^2\right).$$

Remark 2 Suppose we have $\mathbb{Z}$-graded $\mathcal{W}$ superalgebra freely generated with even and odd generators with integral weights $e_1,...,e_k$ and $o_1,...,o_l$ respectively. Then

$$\chi_{\mathcal{W}}(q)=\left(\prod_{m=1}^{k}\prod_{n\geq e_m}\frac{1}{1-q^n}\right)\left(\prod_{m=1}^{l}\prod_{n\geq o_m}(1+q^n)\right).$$

From here we can conclude that the free character and $\chi_{\mathcal{SF}(3)^{S_3}}(q)$ agree up to $O(q^7)$. Therefore generators listed in Theorem 6 are minimal.

Remark 3 . Using methods similar to those above it follows that the orbifold $\mathcal{SF}(3)^{\mathbb{Z}_3}\cong\mathcal{SF}(1)\otimes W$, where W has strong generating set of even vectors

$$\begin{aligned}\widehat{\Omega}^2_{E,E}(0,a)&=E_1(-a-1)E_2\\ \widehat{\Omega}^2_{F,F}(0,a)&=F_1(-a-1)F_2\\ \widehat{\Omega}^{2,1}_{E,F}(0,a)&=E_1(-a-1)F_2\\ \widehat{\Omega}^{2,2}_{E,F}(0,a)&=E_2(-a-1)F_1,\end{aligned}$$

for $a\in\{0,1\}$, and odd vectors

$$\begin{aligned}\widehat{\Omega}^{3,1}_{E,E,F}&=E_1(-a-1)E_1(-1)F_1\\ \widehat{\Omega}^{3,2}_{E,E,F}&=E_2(-a-1)E_2(-1)F_2\\ \widehat{\Omega}^{3,1}_{E,F,F}&=E_1(-1)F_1(-a-1)F_1\\ \widehat{\Omega}^{3,2}_{E,F,F}&=E_2(-1)F_2(-a-1)F_2\end{aligned}$$

for $a \in \{1,2\}$. So, W is of type $2_e^4, 3_e^4, 4_o^4, 5_o^4$ (here e=even and o=odd is the partiy). The character of $\mathcal{SF}(3)^{\mathbb{Z}_3}$ is given by

$$\chi_{\mathcal{SF}(3)^{\mathbb{Z}_3}}(q) = \frac{q^{1/4}}{3}\left(\prod_{n\geq 1}(1+q^n)^6 + 2\prod_{n\geq 1}(1+q^{3n})^2\right).$$

5 Directions for future work

a) In this paper we obtained several different description for the S_3-orbifold algebra of the free neutral fermion. It would interesting to find alternative descriptions for the orbifolds of symplectic fermion vertex algebra.
b) Another interesting direction is to determine the structure of $L_{\mathfrak{sl}_2}(1,0)(3)^{S_3}$ - the S_3-orbifold of the simple level one $\mathfrak{sl}_2$ affine VOA. We can prove that this is an extension of the vertex operator algebra $V_{\sqrt{6}\mathbb{Z}} \otimes L_{Vir}(\frac{4}{5},0) \otimes W_{2,3}^+$, where $V_{\sqrt{6}\mathbb{Z}}$ denotes is the lattice VOA of $\sqrt{6}\mathbb{Z}$, $W_{2,3}^+$ is the fixed point subalgebra of the simple Zamolodchikov $(2,3)$-algebra of central charge $\frac{6}{5}$ under the only non-trivial involution, and $L_{Vir}(c,0)$ is the simple Virasoro vertex algebra of central charge c. More detailed structure will be determined in our future work [**?**].
c) Using the methods of [21] it is possible to determine structure of the S_3 permutation orbifold of the Virasoro vertex algebra. We expect a $\mathcal{W}$-algebra of type $(2,4,5,6^3,7,8^3,9^3,10^2)$ for the $\mathbb{Z}_3$ cyclic orbifold, and of type $(2,4,6^2,8^2,9,10^2,11,12^3,14)$ for the S_3-orbifold. This is work in progress.
d) Everything in this paper can be easily extended to $\mathcal{F}(2)^{D_n}$ and $\mathcal{SF}(2)^{D_n}$ where D_n is dihedral and $n \geq 3$.

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