SOME EXAMPLES OF HIGHER DEPTH VECTOR-VALUED QUANTUM MODULAR FORMS

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Abstract. In this note, we continue our study of generalized quantum modular forms initiated in [4, 5]. We construct further examples of depth two quantum modular forms generalizing several results in [4]. In a special case (corresponding to $p = 2$) we present a more detailed analysis. In particular, a rank two higher depth quantum modular form for the full modular group is constructed.

1. Introduction and statement of results

For $p \in \mathbb{N}$, define the following $\mathfrak{sl}_3$ false theta function

$$F(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{\frac{p}{3}\left(m_1^2+m_2^2+m_1m_2\right)-m_1-m_2+\frac{1}{p}} \left(1-q^{m_1}\right)\left(1-q^{m_2}\right)\left(1-q^{m_1+m_2}\right).$$

This function was introduced in [3] as the numerator of the character of a certain W-algebra associated to $\mathfrak{sl}_3$. A more direct connection between the series and Lie theory can be readily seen from its coefficient $\min(m_1, m_2)$ - the value of Kostant's partition function of $\mathfrak{sl}_3$.

In [4] we decomposed F as

$$F(q) = \frac{2}{p} F_1\left(q^p\right) + 2F_2\left(q^p\right), \tag{1.1}$$

where F_1 and F_2 are generalizations of quantum modular forms. Roughly speaking Zagier [12] defined *quantum modular forms* to be function $f : \mathcal{Q} \to \mathbb{C}$ ($\mathcal{Q} \subset \mathbb{Q}$) such that the "obstruction to modularity"

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \qquad M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$$

is "nice". One can show quantum modular properties of the F_j by using two-dimensional Eichler integrals. For instance, as $\tau \to \frac{h}{k} \in \mathbb{Q}$, F_1 agrees with an integral of the shape ($q := e^{2\pi i\tau}$)

$$\int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f(\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1,$$

where $f \in S_{\frac{3}{2}}(\chi_1, \Gamma) \otimes S_{\frac{3}{2}}(\chi_2, \Gamma)$ (χ_j are certain multipliers and $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$). Throughout we write vectors in bold letters and their components with subscripts. The modular properties of the integral in (1.1) follow from the modularity of f which in turn gives quantum modular properties of F_1. We call the resulting functions higher depth quantum modular forms. Roughly speaking, *depth two quantum modular forms* satisfy, in the simplest case, the modular transformation property with $M = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

$$f(\tau) - (c\tau + d)^{-k} f(M\tau) \in \mathcal{Q}_\kappa(\Gamma)\mathcal{O}(R) + \mathcal{O}(R), \tag{1.2}$$

where $\mathcal{Q}_\kappa(\Gamma)$ is the space of quantum modular forms of weight κ and $\mathcal{O}(R)$ the space of real-analytic functions defined on $R \subset \mathbb{R}$. In [5], we proved that F_1 and F_2 are components of vector-valued quantum modular forms of depth two, generalizing (1.2).

A natural question that arises is what the other components of the vector-valued forms are as q-series. To investigate this, we define, for $1 \leq s_1, s_2 \leq p \in \mathbb{N}$,

$$\mathbb{F}_{\boldsymbol{s}}(q) := \sum_{\substack{m_1,m_2\geq 1\\ m_1\equiv m_2 \pmod 3}} \min(m_1,m_2) q^{\frac{p}{3}\left(\left(m_1-\frac{s_1}{p}\right)^2+\left(m_2-\frac{s_2}{p}\right)^2+\left(m_1-\frac{s_1}{p}\right)\left(m_2-\frac{s_2}{p}\right)\right)}$$
$$\times\left(1-q^{m_1s_1}-q^{m_2s_2}+q^{m_1s_1+(m_1+m_2)s_2}+q^{m_2s_2+(m_1+m_2)s_1}-q^{(m_1+m_2)(s_1+s_2)}\right).$$

Note that $\mathbb{F}_{(1,1)}(q) = F(q)$. As discussed in [3] these series are in fact parametrized by dominant integral weights $(s_1-1)\omega_1 + (s_2-1)\omega_2$ for $\mathfrak{sl}_3$, where ω_j are fundamental weights (dual to simple roots α_1 and α_2).

We may decompose $\mathbb{F}_{\boldsymbol{s}}$ as in (1.1) (see Lemma 2.1). The corresponding functions $\mathbb{F}_{1,\boldsymbol{s}}$ and $\mathbb{F}_{2,\boldsymbol{s}}$ are again generalized quantum modular forms. More precisely, we have.

Theorem 1.1. *The functions $\mathbb{F}_{1,\boldsymbol{s}}$ and $\mathbb{F}_{2,\boldsymbol{s}}$ are depth two quantum modular forms (with respect to some subgroup) of weights one and two, respectively.*

To prove Theorem 1.1, we show that $\mathbb{F}_{1,\boldsymbol{s}}(\tau)$ asymptotically agrees to infinite order with a certain Eichler integral $\mathcal{E}_{1,\boldsymbol{s}}(\frac{\tau}{p})$ defined in (2.1). Similarly, $\mathbb{F}_{2,\boldsymbol{s}}(\tau)$ asymptotically agrees with an Eichler integral $\mathcal{E}_{2,\boldsymbol{s}}(\frac{\tau}{p})$ given in (2.2).

We next restrict to the special case $p = 2$. It turns out (see Lemma 2.2) that for $p = 2$ all $\mathbb{F}_{2,\boldsymbol{s}}$ vanish. Thus we only need to consider $\mathbb{F}_{1,\boldsymbol{s}}$.

Theorem 1.2. *For $p = 2$, the space spanned by $\mathcal{E}_{1,(1,1)}$ and $\mathcal{E}_{1,(1,2)}$ is essentially invariant under modular transformations. By this we mean that the only terms appearing in the modular transformations which do not lie in the space are simpler (see* (2.6) *and* (2.7) *for the case of inversion).*

Motivated by representation theory of the W-algebra $W^0(p)_{A_2}$ studied in [3, 8], we raise the following.

Conjecture. *After multiplication with η^2, the characters of $W^0(p)_{A_2}$ given in* [3, Section 5] *(which also includes the series $\mathbb{F}_{\boldsymbol{s}}$) combine into a vector-valued quantum modular form of depth two.*

The second goal of this paper is to determine the asymptotic behavior of $\mathcal{E}_{1,\boldsymbol{s}}(it)$ as $t \to 0^+$. It is well-known that asymptotic behaviors of vector-valued modular forms (as $t \to 0^+$) can be computed by applying the S-transformation $\tau \mapsto -\frac{1}{\tau}$, and then analyzing the dominating term. This method is widely used for studying quantum dimensions of modules of vertex algebras (and affine Lie algebras) as their characters often transform invariantly under $\mathrm{SL}_2(\mathbb{Z})$. In this paper we work with functions (coming also from characters) that transform with higher depth error terms so their asymptotics are more interesting and harder to analyze. We show that asymptotic behavior of double Eichler integrals can be also analyzed by using a similar approach. We do this directly from the integral representation of the error function. In the body of the paper, we show that it is enough to study

$$\mathbb{E}_{1,(1,1)}(\tau) := 4I_{(1,3)}(\tau) \quad \text{and} \quad \mathbb{E}_{1,(1,2)}(\tau) := 2I_{(1,1)}(\tau) + 2I_{(1,5)}(\tau), \tag{1.3}$$

where the theta integrals $I_{\boldsymbol{k}}$ are defined in (2.3). We prove the following.

Theorem 1.3. *We have, as $t \to 0^+$,*

$$\mathbb{E}_{1,(1,1)}(it) \sim \frac{1}{4}, \qquad \mathbb{E}_{1,(1,2)}(it) \sim \frac{3}{4}.$$

Note that the asymptotics in Theorem 1.3 agree with the answer which one obtains from [5] by using two-dimensional false theta functions.

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2. Proof of Theorem 1.1 and Theorem 1.2

To prove Theorem 1.1 and Theorem 1.2, we let

$$\mathbb{F}_{1,\boldsymbol{s}}(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}} \varepsilon_{\boldsymbol{s}}(\alpha) \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} q^{pQ(\boldsymbol{n}+\boldsymbol{\alpha})},$$

where $Q(x_1,x_2) := 3x_1^2+3x_1x_2+x_2^2$ and where

$$\begin{aligned}\mathscr{S}_{\boldsymbol{s}} := \Bigg\{ &\left(\frac{s_2-s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_2-s_1}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1+s_2}{p}\right),\\ &\left(\frac{2s_2+s_1}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_2}{p}\right), \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1}{p}\right),\\ &\left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_1+s_2}{p}\right),\\ &\left(1-\frac{s_2+2s_1}{3p}, \frac{s_1+s_2}{p}\right), \left(\frac{s_2-s_1}{3p}, \frac{s_1}{p}\right), \left(1-\frac{s_2-s_1}{3p}, \frac{s_2}{p}\right)\Bigg\},\end{aligned}$$

$$\varepsilon_{\boldsymbol{s}}(\boldsymbol{\alpha}) := \begin{cases} s_2 & \text{if } \boldsymbol{\alpha} \in \left\{ \left(\frac{s_2-s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_2}{p}\right), \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_2}{p}\right), \right.\\ & \qquad\qquad \left.\left(1-\frac{s_2-s_1}{3p}, \frac{s_2}{p}\right)\right\},\\ s_1 & \text{if } \boldsymbol{\alpha} \in \left\{ \left(1-\frac{s_2-s_1}{3p}, 1-\frac{s_1}{p}\right), \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1}{p}\right), \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1}{p}\right), \right.\\ & \qquad\qquad \left.\left(\frac{s_2-s_1}{3p}, \frac{s_1}{p}\right)\right\},\\ -(s_1+s_2) & \text{if } \boldsymbol{\alpha} \in \left\{ \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_1+s_2}{p}\right), \right.\\ & \qquad\qquad \left.\left(1-\frac{s_2+2s_1}{3p}, \frac{s_1+s_2}{p}\right)\right\} \end{cases}$$

and

$$\mathbb{F}_{2,\boldsymbol{s}}(q) := \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}} \eta_{\boldsymbol{s}}(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2)\, q^{Q(\boldsymbol{n}+\boldsymbol{\alpha})},$$

where

$$\eta_{\boldsymbol{s}}(\boldsymbol{\alpha}) := \begin{cases} 1 & \text{if } \boldsymbol{\alpha} \in \left\{ \left(\frac{s_2-s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_2-s_1}{3p}, 1-\frac{s_1}{p}\right), \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1}{p}\right), \right.\\ & \qquad \left.\left(\frac{2s_2+s_1}{3p}, 1-\frac{s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_1+s_2}{p}\right), \left(1-\frac{s_2+2s_1}{3p}, \frac{s_1+s_2}{p}\right)\right\},\\ -1 & \text{if } \boldsymbol{\alpha} \in \left\{ \left(\frac{2s_1+s_2}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(\frac{2s_2+s_1}{3p}, 1-\frac{s_1+s_2}{p}\right), \left(1-\frac{s_1+2s_2}{3p}, \frac{s_2}{p}\right), \right.\\ & \qquad \left.\left(1-\frac{s_2+2s_1}{3p}, \frac{s_1}{p}\right), \left(\frac{s_2-s_1}{3p}, \frac{s_1}{p}\right), \left(1-\frac{s_2-s_1}{3p}, \frac{s_2}{p}\right)\right\}. \end{cases}$$

Remark. We have
$$\mathbb{F}_{(p,p)}(q)=1.$$
Thus we may throughout assume that $\boldsymbol{s}\neq(p,p)$.

Similarly as in the case $\boldsymbol{s}=(1,1)$, a lengthy calculation gives.

Lemma 2.1. *We have*
$$\mathbb{F}_{\boldsymbol{s}}(q)=\frac{1}{p}\mathbb{F}_{1,\boldsymbol{s}}\left(q^p\right)+\mathbb{F}_{2,\boldsymbol{s}}\left(q^p\right).$$

The following theorem states quantum modular properties of the functions $\mathbb{F}_{1,\boldsymbol{s}}$ and $\mathbb{F}_{2,\boldsymbol{s}}$, using the method of [4]. Let
$$\mathcal{E}_{1,\boldsymbol{s}}(\tau):=\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*}\varepsilon_{\boldsymbol{s}}(\boldsymbol{\alpha})\mathcal{E}_{1,\boldsymbol{\alpha}}(p\tau),\tag{2.1}$$
where
$$\begin{aligned}\mathscr{S}_{\boldsymbol{s}}^*:=\Bigg\{&\left(\frac{s_2-s_1}{3p},1-\frac{s_2}{p}\right),\left(1-\frac{s_2-s_1}{3p},1-\frac{s_1}{p}\right),\left(\frac{2s_1+s_2}{3p},1-\frac{s_1}{p}\right),\\ &\left(\frac{2s_2+s_1}{3p},1-\frac{s_2}{p}\right),\left(1-\frac{s_1+2s_2}{3p},\frac{s_1+s_2}{p}\right)\left(1-\frac{s_2+2s_1}{3p},\frac{s_1+s_2}{p}\right)\Bigg\}.\end{aligned}$$
Moreover, the Eichler integrals $\mathcal{E}_{1,\boldsymbol{\alpha}}$ are given as
$$\mathcal{E}_{1,\boldsymbol{\alpha}}(\tau):=-\frac{\sqrt{3}}{4}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\theta_1(\boldsymbol{\alpha};\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1$$
with
$$\begin{aligned}\theta_1(\boldsymbol{\alpha};\boldsymbol{w})&:=\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}(2n_1+n_2)n_2e^{\frac{3\pi i}{2}(2n_1+n_2)^2w_1+\frac{\pi i n_2^2w_2}{2}},\\ \theta_2(\boldsymbol{\alpha};\boldsymbol{w})&:=\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}(3n_1+2n_2)n_1e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i n_1^2w_2}{2}}.\end{aligned}$$
Finally let
$$\mathcal{E}_{2,\boldsymbol{s}}(\tau):=\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*}\mathcal{E}_{2,\boldsymbol{\alpha}}(p\tau).\tag{2.2}$$
Here
$$\begin{aligned}\mathcal{E}_{2,\boldsymbol{\alpha}}(\tau):=&\frac{\sqrt{3}}{8\pi}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{2\theta_3(\boldsymbol{\alpha};\boldsymbol{w})-\theta_4(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}(-i(w_2+\tau))^{\frac{3}{2}}}dw_2dw_1\\ &+\frac{\sqrt{3}}{8\pi}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\theta_5(\boldsymbol{\alpha};\boldsymbol{w})}{(-i(w_1+\tau))^{\frac{3}{2}}\sqrt{-i(w_2+\tau)}}dw_2dw_1\end{aligned}$$
with
$$\begin{aligned}\theta_3(\boldsymbol{\alpha};\boldsymbol{w})&:=\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}(2n_1+n_2)e^{\frac{3\pi i}{2}(2n_1+n_2)^2w_1+\frac{\pi i n_2^2w_2}{2}},\\ \theta_4(\boldsymbol{\alpha};\boldsymbol{w})&:=\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}(3n_1+2n_2)e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i n_1^2w_2}{2}},\\ \theta_5(\boldsymbol{\alpha};\boldsymbol{w})&:=\sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2}n_1e^{\frac{\pi i}{2}(3n_1+2n_2)^2w_1+\frac{3\pi i n_1^2w_2}{2}}.\end{aligned}$$

Furthermore define, for $\nu \in \{0,1\}$, $h \in \mathbb{Z}$, $N, A \in \mathbb{N}$ with $A|N$ and $N|hA$, the theta function studied, for example, by Shimura [11]

$$\Theta_\nu(A,h,N;\tau) := \sum_{\substack{m\in\mathbb{Z}\\ m\equiv h \pmod{\mathrm{N}}}} m^\nu q^{\frac{Am^2}{2N^2}}.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 (Sketch): We start with $\mathbb{F}_{1,\boldsymbol{s}}$. Write

$$\mathbb{F}_{1,\boldsymbol{s}}\left(e^{2\pi i\frac{h}{k}-t}\right) \sim \sum_{m\geq 0} A_{\boldsymbol{s},h,k}(m)t^m \quad \left(t\to 0^+\right).$$

Using the Euler-Maclaurin summation formula (in the shape stated in (28) of [4]) one can prove, following the proof of Theorem 7.1 of [4], that

$$\mathbb{E}_{1,\boldsymbol{s}}\left(\frac{it}{2\pi}-\frac{h}{k}\right) \sim \sum_{m\geq 0} A_{\boldsymbol{s},h,k}(m)(-t)^m \quad \left(t\to 0^+\right).$$

Here

$$\mathbb{E}_{1,\boldsymbol{s}}(\tau) := \frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*} \varepsilon_{\boldsymbol{s}}(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} M_2\left(\sqrt{3};\sqrt{v}\left(2\sqrt{3}n_1+\sqrt{3}n_2,n_2\right)\right) q^{-Q(\boldsymbol{n})},$$

where $\boldsymbol{w}\in\mathbb{R}^2$ and $\kappa\in\mathbb{R}$ with $w_2,\ w_1-\kappa w_2\neq 0$, we set

$$M_2(\kappa;\boldsymbol{w}) := -\frac{1}{\pi^2}\int_{\mathbb{R}^2-i\boldsymbol{w}} \frac{e^{-\pi t_1^2-\pi t_2^2-2\pi i(t_1w_1+t_2w_2)}}{t_2(t_2-\kappa t_1)}\, dt_1dt_2.$$

In particular, $\mathbb{E}_{1,s}$ agrees with $\mathbb{F}_{1,s}$ on $\mathbb{Q}$. Proceeding as in the proof of Lemma 6.1 of [4] one can then show that

$$\mathbb{E}_{1,\boldsymbol{s}}(\tau) = \mathcal{E}_{1,\boldsymbol{s}}\left(\frac{\tau}{p}\right).$$

To determine the transformation behaviour, we rewrite the theta functions in $\mathcal{E}_{1,\boldsymbol{s}}$ in terms of Shimura theta functions to obtain, as in the proof of Proposition 5.2 of [4]

$$3p\mathcal{E}_{1,\boldsymbol{s}}\left(\frac{\tau}{p}\right) = (2s_1+s_2)J_{(s_2,s_2+2s_1)}(\tau) + (2s_2+s_1)J_{(s_1,s_1+2s_2)}(\tau) + (s_2-s_1)J_{(s_1+s_2,s_1-s_2)}(\tau),$$

where

$$J_{\boldsymbol{k}}(\tau) := \sum_{\delta\in\{0,1\}} I_{(k_1+\delta p,k_2+3\delta p)}(\tau) \quad \text{with} \quad I_{\boldsymbol{k}}(\tau) := -\frac{\sqrt{3}}{4p} I_{\Theta_1(2p,k_1,2p;\cdot),\Theta_1(6p,k_2,6p,\cdot)}(\tau). \tag{2.3}$$

Here, for modular forms f and g of weights κ_1 and κ_2, respectively,

$$I_{f,g}(\tau) := \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty} \frac{f(w_1)g(w_2)}{(-i(w_1+\tau))^{2-\kappa_1}(-i(w_2+\tau))^{2-\kappa_2}} dw_2dw_1.$$

Now the transformation properties follow as in the proof of Proposition 5.2 of [5].

For the function $\mathbb{F}_{2,\boldsymbol{s}}$, we proceed in the same way. Writing

$$\mathbb{F}_{2,\boldsymbol{s}}\left(e^{2\pi i\frac{h}{k}-t}\right) \sim \sum_{m\geq 0} B_{\boldsymbol{s},h,k}(m)t^m \quad \left(t\to 0^+\right)$$

we may show in a similar manner as in the proof of Theorem 7.2 of [4], using the Euler-Maclaurin summation formula, that

$$\mathbb{E}_{2,\boldsymbol{s}}\left(\frac{it}{2\pi}-\frac{h}{k}\right) \sim \sum_{m\geq 0} B_{\boldsymbol{s},h,k}(m)(-t)^m.$$

Here

$$\mathbb{E}_2(\tau) = \mathbb{E}_{2,\boldsymbol{s}}(\tau)$$
$$:= \frac{1}{4\pi i} \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*} \sum_{\boldsymbol{n}\in\boldsymbol{\alpha}+\mathbb{Z}^2} \left[\frac{\partial}{\partial z}\left(M_2\left(\sqrt{3}; \sqrt{3v}(2n_1+n_2), \sqrt{v}\left(n_2 - \frac{2\,\mathrm{Im}(z)}{v}\right)\right) e^{2\pi i n_2 z}\right)\right]_{z=0} q^{-Q(\boldsymbol{n})}.$$

Following the proof of Lemma 6.2 of [4], one may then prove that

$$\mathbb{E}_{2,\boldsymbol{s}}(\tau) = \mathcal{E}_{2,\boldsymbol{s}}\left(\frac{\tau}{p}\right).$$

To finish the proof one may show that, proceeding as in the proof of Proposition 5.2 of [4].

$$\mathbb{E}_{2,\boldsymbol{s}}(\tau) = \frac{2}{p}\left(-\mathcal{J}_{(s_1+s_2,s_1-s_2)}(\tau) + \mathcal{J}_{(s_2,2s_1+s_2)}(\tau) + \mathcal{J}_{(s_1,2s_2+s_1)}(\tau)\right),$$

where

$$\mathcal{J}_{\boldsymbol{k}}(\tau) := \sum_{\delta\in\{0,1\}} \mathcal{I}_{(k_1+p\delta,k_2+3p\delta)}(\tau), \quad \text{with} \quad \mathcal{I}_{\boldsymbol{k}}(\tau) := -\frac{\sqrt{3}}{8\pi} I_{\Theta_1(2p,k_1,2p;\cdot),\Theta_0(6p,k_2,6p;\cdot)}(\tau).$$

Again the transformation properties follow as in the proof of Proposition 5.5 of [5]. □

We now restrict to $p=2$. The following lemma shows the vanishing of $\mathbb{F}_{2,\boldsymbol{s}}$ in this case.

Lemma 2.2. *For $p=2$, the functions $\mathbb{F}_{2,\boldsymbol{s}}$ and $\mathbb{E}_{2,\boldsymbol{s}}$ vanish identically.*

Proof: We start by proving that $\mathbb{F}_{2,\boldsymbol{s}}=0$. It is enough to consider $\boldsymbol{s}\in\{(1,1),(1,2)\}$. The claim for $\boldsymbol{s}=(1,1)$ follows directly by plugging in the definition of $\mathbb{F}_{2,(1,1)}$ and canceling terms.

We next consider $\mathbb{F}_{2,(1,2)}$. By definition

$$\mathbb{F}_{2,(1,2)}(q) = \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,2)}} \eta_{(1,2)}(\boldsymbol{\alpha}) \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2)\, q^{Q(\boldsymbol{n}+\boldsymbol{\alpha})},$$

where

$$\eta_{(1,2)}(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha}\in\left\{\left(\frac{1}{6},0\right),\left(\frac{5}{6},\frac{1}{2}\right),\left(\frac{2}{3},\frac{1}{2}\right),\left(\frac{5}{6},0\right),\left(\frac{1}{6},\frac{3}{2}\right),\left(\frac{1}{3},\frac{3}{2}\right)\right\}, \\ -1 & \text{if } \boldsymbol{\alpha}\in\left\{\left(\frac{2}{3},-\frac{1}{2}\right),\left(\frac{5}{6},-\frac{1}{2}\right),\left(\frac{1}{6},1\right),\left(\frac{1}{3},\frac{1}{2}\right),\left(\frac{1}{6},\frac{1}{2}\right),\left(\frac{5}{6},1\right)\right\}. \end{cases}$$

Note that

$$H_{\boldsymbol{\alpha}}(q) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2)\, q^{Q(\boldsymbol{n}+\boldsymbol{\alpha})} - \sum_{\boldsymbol{n}\in\mathbb{N}_0^2} (n_2+\alpha_2-1)\, q^{Q(\boldsymbol{n}+(\alpha_1,\alpha_2-1))}$$
$$= (1-\alpha_2)\, q^{\frac{1}{4}(\alpha_2-1)^2} \sum_{n\in\alpha_1+\frac{\alpha_2-1}{2}+\mathbb{N}_0} q^{3n^2}.$$

Thus

$$\mathbb{F}_{2,(1,2)}(q) = -H_{\left(\frac{1}{6},1\right)}(q) + H_{\left(\frac{5}{6},\frac{1}{2}\right)}(q) + H_{\left(\frac{2}{3},\frac{1}{2}\right)}(q) - H_{\left(\frac{5}{6},1\right)}(q) + H_{\left(\frac{1}{6},\frac{3}{2}\right)}(q) + H_{\left(\frac{1}{3},\frac{3}{2}\right)}(q)$$
$$= \frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{7}{12}+\mathbb{N}_0} q^{3n^2} + \frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{5}{12}+\mathbb{N}_0} q^{3n^2} - \frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{5}{12}+\mathbb{N}_0} -\frac{1}{2}q^{\frac{1}{16}} \sum_{n\in\frac{7}{12}+\mathbb{N}_0} q^{3n^2} = 0.$$

To see that $\mathbb{E}_{2,\mathbf{s}}=0$, it is sufficient to prove

$$-\mathcal{J}_{(s_1+s_2,s_1-s_2)} + \mathcal{J}_{(s_2,2s_1+s_2)} + \mathcal{J}_{(s_1,2s_2+s_1)} = 0,$$

which is a straightforward computation with theta series. □

We are now ready to prove Theorem 1.2.

Sketch of proof of Theorem 1.2: We write

$$\mathbb{E}_{1,\boldsymbol{s}}(\tau) = -\frac{\sqrt{3}}{2}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*}\varepsilon(\boldsymbol{\alpha})\left(\theta_1(\boldsymbol{\alpha};2\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};2\boldsymbol{w})\right)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1.$$

We next show the identities in (1.3). We start with $\boldsymbol{s}=(1,1)$. We use the theta relation

$$\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,1)}^*}\varepsilon(\boldsymbol{\alpha})\left(\theta_1(\boldsymbol{\alpha};2\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};2\boldsymbol{w})\right)=\frac{1}{2}\Theta_1(4,1,4;w_1)\Theta_1(12,3,12;w_2). \tag{2.4}$$

Equation (2.4) yields

$$\mathbb{E}_{1,(1,1)}(\tau) = -\frac{\sqrt{3}}{2}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\Theta_1(4,1,4;w_1)\Theta_1(12,3,12;w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1 = 4I_{(1,3)}(\tau),$$

which is the first identity in (1.3).

We next consider $\mathbb{E}_{1,(1,2)}$ and use that

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,2)}^*}\varepsilon(\boldsymbol{\alpha})\left(\theta_1(\boldsymbol{\alpha};2\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};2\boldsymbol{w})\right)=\frac{1}{2}\Theta_1(4,1,4;w_1)\left(\Theta_1(12,1,12;w_2)+\Theta_1(12,5,12;w_2)\right). \tag{2.5}$$

Thus

$$\begin{aligned}\mathbb{E}_{1,(1,2)}(\tau) &= -\frac{\sqrt{3}}{4}\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\Theta_1(4,1,4;w_1)\left(\Theta_1(12,1,12;w_2)+\Theta_1(12,5,12;w_2)\right)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1\\ &= 2(I_{(1,1)}(\tau)+I_{(1,5)}(\tau)),\end{aligned}$$

which is the second identity in (1.3).

We next use Lemma 5.1 of [5], to obtain

$$I_{\boldsymbol{k}}(\tau) = (-i\tau)^{-1}\frac{1}{\sqrt{3}}\sum_{k=1}^{5}\sin\left(\frac{\pi k k_2}{6}\right)I_{(k_1,k)}\left(-\frac{1}{\tau}\right)+\mathbb{A}_{\boldsymbol{k}}(\tau),$$

where $\mathbb{A}_{\boldsymbol{k}}$ contributes the simpler terms mentioned in Theorem 1.2, and is explicitly given by

$$\begin{aligned}\mathbb{A}_{\boldsymbol{k}}(\tau) &:= -\frac{\sqrt{3}}{8}\int_{0}^{i\infty}\int_{w_1}^{i\infty}\frac{\Theta_1(4,k_1,4;w_1)\Theta_1(12,k_2,12;w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1\\ &\quad-\frac{\sqrt{3}}{8}I_{\Theta_1(4,k_1,4;\cdot)}(\tau)r_{\Theta_1(12,k_2,12;\cdot)}(\tau)+\frac{\sqrt{3}}{8}r_{\Theta_1(4,k_1,4;\cdot)}(\tau)r_{\Theta_1(12,k_2,12;\cdot)}(\tau),\end{aligned}$$

where, for f a holomorphic modular form of weight k,

$$r_f(\tau) := \int_0^{i\infty} f(w)(-i(w+\tau))^{k-2}dw.$$

In particular

$$\begin{aligned}\mathbb{E}_{1,(1,1)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(2\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right)-\mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right)+4\mathbb{A}_{(1,3)}(\tau),\\ \mathbb{E}_{1,(1,2)}(\tau) &= \frac{1}{\sqrt{3}(-i\tau)}\left(\mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)+\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right)\right)+2\mathbb{A}_{(1,1)}(\tau)+2\mathbb{A}_{(1,5)}(\tau).\end{aligned}$$

Inverting and reordering gives

$$\mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right) = -\frac{i\tau}{\sqrt{3}}\big(2\mathbb{E}_{1,(1,2)}(\tau)-\mathbb{E}_{1,(1,1)}(\tau)\big)-\frac{4i\tau}{\sqrt{3}}\left(\mathbb{A}_{(1,3)}(\tau)-\mathbb{A}_{(1,1)}(\tau)-\mathbb{A}_{(1,5)}(\tau)\right), \tag{2.6}$$

$$\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right)=-\frac{i\tau}{\sqrt{3}}\left(\mathbb{E}_{1,(1,2)}(\tau)+\mathbb{E}_{1,(1,1)}(\tau)\right)+\frac{2i\tau}{\sqrt{3}}\left(\mathbb{A}_{(1,1)}(\tau)+\mathbb{A}_{(1,5)}(\tau)+2\mathbb{A}_{(1,3)}(\tau)\right). \tag{2.7}$$

The claim follows using that

$$\mathbb{E}_{1,(1,1)}(\tau+1)=-\mathbb{E}_{1,(1,1)}(\tau),\quad \mathbb{E}_{1,(1,2)}(\tau+1)=e^{-\frac{\pi i}{6}}\mathbb{E}_{1,(1,2)}(\tau).$$

□

3. The asymptotic behavior of $H_{1,\boldsymbol{\alpha}}$

To prove Theorem 1.3 we need to compute

$$H_{\boldsymbol{\alpha}}:=\lim_{t\to 0^+}\frac{H_{1,\boldsymbol{\alpha}}\left(\frac{i}{t}\right)}{t},$$

where, for $\boldsymbol{\alpha}\in\mathbb{R}^2$,

$$H_{1,\boldsymbol{\alpha}}(\tau):=-\sqrt{3}\int_0^{i\infty}\int_{w_1}^{i\infty}\frac{\theta_1(\boldsymbol{\alpha};\boldsymbol{w})+\theta_2(\boldsymbol{\alpha};\boldsymbol{w})}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1.$$

Proposition 3.1. *Assume that α_1,α_2 are not both in $\mathbb{Z}$. We have*

$$H_{\boldsymbol{\alpha}}=\begin{cases}\frac{2}{\sqrt{3}}\frac{\sin(2\pi\alpha_1)\sin(2\pi\alpha_2)}{(1-\cos(2\pi\alpha_1))(1-\cos(2\pi\alpha_2))} & \text{if } \alpha_1,\alpha_2\notin\mathbb{Z},\\ \frac{2\sqrt{3}}{1-\cos(2\pi\alpha_2)} & \text{if } \alpha_1\in\mathbb{Z},\alpha_2\notin\mathbb{Z},\\ \frac{2}{(1-\cos(2\pi\alpha_1))\sqrt{3}} & \text{if } \alpha_1\notin\mathbb{Z},\alpha_2\in\mathbb{Z}.\end{cases}$$

Proof: We first rewrite $H_{1,\boldsymbol{\alpha}}(\tau)$. By Theorem 1.2 of [5], we have

$$H_{1,\boldsymbol{\alpha}}(\tau)=\int_{\mathbb{R}^2}g_{1,\boldsymbol{\alpha}}(\boldsymbol{w})e^{2\pi i\tau Q(\boldsymbol{w})}dw_1dw_2.$$

Here we define

$$g_{1,\boldsymbol{\alpha}}(\boldsymbol{w}):=\begin{cases}2\mathcal{G}_{\alpha_1}(w_1)\mathcal{G}_{\alpha_2}(w_2)-2\mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_{\alpha_2}(w_2) & \text{if } \alpha_1,\alpha_2\notin\mathbb{Z},\\ -2\mathcal{F}_0(w_1)\mathcal{F}_{\alpha_2}(w_2)+\frac{2}{\pi w_1}\mathcal{F}_{\alpha_2}\left(w_2+\frac{3w_1}{2}\right) & \text{if } \alpha_1\in\mathbb{Z},\ \alpha_2\notin\mathbb{Z},\\ -2\mathcal{F}_{\alpha_1}(w_1)\mathcal{F}_0(w_2)+\frac{2}{\pi w_2}\mathcal{F}_{\alpha_1}\left(w_1+\frac{w_2}{2}\right) & \text{if } \alpha_1\notin\mathbb{Z},\ \alpha_2\in\mathbb{Z},\end{cases}$$

setting

$$\mathcal{F}_\alpha(x):=\frac{\sinh(2\pi x)}{\cosh(2\pi x)-\cos(2\pi\alpha)},\qquad \mathcal{G}_\alpha(x):=\frac{\sin(2\pi\alpha)}{\cosh(2\pi x)-\cos(2\pi\alpha)}.$$

Applying the two-dimensional saddle point method gives that

$$H_{\boldsymbol{\alpha}}=\frac{g_{1,\boldsymbol{\alpha}}(0,0)}{\sqrt{3}}.$$

Explicitly computing $g_{1,\boldsymbol{\alpha}}(0,0)$ yields the claim of Proposition 3.1.

□

4. Proof of Theorem 1.3.

Inverting (2.6) and (2.7) gives

$$\mathbb{E}_{1,(1,1)}(\tau)=\frac{1}{\sqrt{3}(-i\tau)}\left(2\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right)-\mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right)$$
$$+\frac{4}{\sqrt{3}(-i\tau)}\left(\mathbb{A}_{(1,3)}\left(-\frac{1}{\tau}\right)-\mathbb{A}_{(1,1)}\left(-\frac{1}{\tau}\right)-\mathbb{A}_{(1,5)}\left(-\frac{1}{\tau}\right)\right),$$

$$
\begin{aligned}
\mathbb{E}_{1,(1,2)}(\tau) = \frac{1}{\sqrt{3}(-i\tau)} &\left(\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right) + \mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right) \\
&- \frac{2}{\sqrt{3}(-i\tau)}\left(\mathbb{A}_{(1,1)}\left(-\frac{1}{\tau}\right) + \mathbb{A}_{(1,5)}\left(-\frac{1}{\tau}\right) + 2\mathbb{A}_{(1,3)}\left(-\frac{1}{\tau}\right)\right).
\end{aligned}
$$

We next rewrite the first summand of $\mathbb{A}_{(1,j)}$, denoting it by $\mathbb{B}_{(1,j)}$. For this, we again use the theta relations (2.4) and (2.5). This yields

$$
\mathbb{B}_{(1,3)}(\tau) = \frac{1}{16}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha}) H_{1,\boldsymbol{\alpha}}(2\tau), \qquad \mathbb{B}_{(1,1)}(\tau) + \mathbb{B}_{(1,5)}(\tau) = \frac{1}{8}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha}) H_{1,\boldsymbol{\alpha}}(2\tau).
$$

Thus

$$
\begin{aligned}
\mathbb{E}_{1,(1,1)}(\tau) = {}& \frac{1}{\sqrt{3}(-i\tau)}\left(2\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right) - \mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right) \\
&+ \frac{1}{2\sqrt{3}(-i\tau)}\left(\frac{1}{2}\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha}) H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right) - \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha}) H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right)\right) \\
&- \frac{1}{2(-i\tau)}\left(I_{\Theta_1(4,1,4)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(4,1,4)}\left(-\frac{1}{\tau}\right)\right) \\
&\qquad \times \left(r_{\Theta_1(12,3,12)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(12,1,12)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(12,5,12)}\left(-\frac{1}{\tau}\right)\right), \\
\mathbb{E}_{1,(1,2)}(\tau) = {}& \frac{1}{\sqrt{3}(-i\tau)}\left(\mathbb{E}_{1,(1,2)}\left(-\frac{1}{\tau}\right) + \mathbb{E}_{1,(1,1)}\left(-\frac{1}{\tau}\right)\right) \\
&- \frac{1}{4\sqrt{3}(-i\tau)}\left(\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha}) H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right) + \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha}) H_{1,\boldsymbol{\alpha}}\left(-\frac{2}{\tau}\right)\right) \\
&+ \frac{1}{4(-i\tau)}\left(I_{\Theta_1(4,1,4,1)}\left(-\frac{1}{\tau}\right) - r_{\Theta_1(4,1,4)}\left(-\frac{1}{\tau}\right)\right. \\
&\qquad \left.\times \left(2r_{\Theta_1(12,3,12;\,\cdot\,)}\left(-\frac{1}{\tau}\right) + r_{\Theta_1(12,1,12;\,\cdot\,)}\left(-\frac{1}{\tau}\right) + r_{\Theta_1(12,5,12;\,\cdot\,)}\left(-\frac{1}{\tau}\right)\right)\right).
\end{aligned}
$$

Letting $\tau = it \to 0$ yields

$$
\mathbb{E}_{1,(1,1)}(it) \sim \frac{1}{8\sqrt{3}}\left(\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}} - 2\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}\right) + \frac{1}{2}(h_3 - h_1 - h_5), \tag{4.1}
$$

$$
\mathbb{E}_{1,(1,2)}(it) \sim -\frac{1}{8\sqrt{3}}\left(\sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,1)}} \varepsilon(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}} + \sum_{\boldsymbol{\alpha}\in\mathscr{S}^*_{(1,2)}} \varepsilon(\boldsymbol{\alpha}) H_{\boldsymbol{\alpha}}\right) - \frac{1}{4}(2h_3 + h_1 + h_5), \tag{4.2}
$$

where

$$
h_j := \lim_{t\to 0} \frac{1}{t} r_{\Theta_1(4,1,4;\,\cdot\,)}\left(\frac{i}{t}\right) r_{\Theta_1(12,j,12;\,\cdot\,)}\left(\frac{i}{t}\right).
$$

We have

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{\boldsymbol{s}}^*}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} = s_2 H_{\left(\frac{s_2-s_1}{6},1-\frac{s_2}{2}\right)} + s_1 H_{\left(1-\frac{s_2-s_1}{6},1-\frac{s_1}{2}\right)} + s_1 H_{\left(\frac{2s_1+s_2}{6},1-\frac{s_1}{2}\right)} + s_2 H_{\left(\frac{2s_2+s_1}{6},1-\frac{s_2}{2}\right)}$$
$$- (s_1+s_2) H_{\left(1-\frac{s_1+2s_2}{6},\frac{s_1+s_2}{2}\right)} - (s_1+s_2) H_{\left(1-\frac{s_2+2s_1}{6},\frac{s_1+s_2}{2}\right)}.$$

In particular, using Proposition 1.1, we evaluate

$$\sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,1)}^*}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} = \frac{2}{\sqrt{3}}, \qquad \sum_{\boldsymbol{\alpha}\in\mathscr{S}_{(1,2)}^*}\varepsilon(\boldsymbol{\alpha})H_{\boldsymbol{\alpha}} = \frac{16}{\sqrt{3}}. \tag{4.3}$$

To compute $\lim_{t\to 0} t^{-\frac{1}{2}} r_{\Theta_1(N,a,N;\,\cdot\,)}(\frac{i}{t})$ we employ Lemma 3.2 of [5] to obtain

$$r_{\Theta_1(N,a,N;\,\cdot\,)}\left(\frac{i}{t}\right) = \frac{i\sqrt{N}}{2}\sin\left(\frac{2\pi a}{N}\right)\int_{\mathbb{R}}\frac{e^{-\frac{\pi N}{t}x^2}}{\sinh\left(\pi x+\frac{\pi i a}{N}\right)\sinh\left(\pi x-\frac{\pi i a}{N}\right)}dx.$$

The saddle point method then yields that

$$r_{\Theta_1(N,a,N;\,\cdot\,)}\left(\frac{i}{t}\right) = i\sqrt{t}\cot\left(\frac{\pi a}{N}\right).$$

Thus

$$h_j = \cot\left(\frac{\pi j}{12}\right).$$

In particular

$$h_1 = -\cot\left(\frac{\pi}{12}\right), \qquad h_3 = -1, \qquad h_5 = -\cot\left(\frac{5\pi}{12}\right).$$

Plugging this and (4.3) into (4.1) and (4.2) gives the claim.

5. Simplification for $p=2$

We first recall the one-dimensional situation for $p=2$. There is a unique false theta function

$$\sum_{n\in\mathbb{Z}}\operatorname{sgn}\left(n+\frac{1}{2}\right)q^{2\left(n+\frac{1}{4}\right)^2},$$

whose corresponding Eichler integral is (see [3])

$$F_{1,2}^*(\tau) := -2i\int_{-\overline{\tau}}^{i\infty}\frac{\Theta_1(4,1,4;w)}{\sqrt{-i(w+\tau)}}dw.$$

Noting that

$$\Theta_1(4,1,4;\tau) = \eta(\tau)^3, \tag{5.1}$$

this integral transforms as a scalar-valued quantum modular form of weight $\frac{1}{2}$.

In the two-dimensional case, a similar "higher depth" picture emerges. Observing (5.1) and

$$\Theta_1(12,3,12;\tau) = 3\eta(3\tau)^3, \quad \Theta_1(12,1,12;\tau)+\Theta_1(12,5,12;\tau) = 3\eta(3\tau)^3+\eta\left(\frac{\tau}{3}\right)^3$$

we obtain that the space spanned by $\mathbb{E}_{1,(1,1)}(\tau)$ and $\mathbb{E}_{1,(1,2)}(\tau)$ is also spanned by

$$\int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\eta(w_1)^3\eta(3w_2)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1, \quad \int_{-\overline{\tau}}^{i\infty}\int_{w_1}^{i\infty}\frac{\eta(w_1)^3\eta\left(\frac{w_2}{3}\right)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}}dw_2dw_1. \tag{5.2}$$

The next result can be found in [10, Corollary 6.6] (it can be also derived by using representation theory of $\widehat{\mathfrak{sl}}_3$ as discussed in [2]).

Proposition 5.1. *We have*

$$\eta(\tau)\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn} = 3\eta(3\tau)^3 + \eta\left(\frac{\tau}{3}\right)^3, \quad \eta(\tau)q^{\frac{1}{3}}\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn+n} = 3\eta(3\tau)^3.$$

According to [9], $\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn}$ and $q^{\frac{1}{3}}\sum_{m,n\in\mathbb{Z}} q^{m^2+n^2-mn+n}$ are numerators of two characters of irreducible highest weight $\widehat{\mathfrak{sl}}_3$-modules of level one. Therefore modular properties of the double Eichler integrals in (5.2), modulo correction factors, are identical to modular transformation properties of the span of characters of the level one simple affine vertex algebra of $\widehat{\mathfrak{sl}}_3$. It would be interesting to understand a possible connection from a purely representation theoretic perspective. This is closely related to a conjecture of Creutzig and the third author [8] pertaining to quantum modular properties of characters of $W^0(p)_{A_2}$, representations of affine Lie algebras, and representations of quantum groups at a root of unity (see also [1, 6, 7] for other appearances of this and related vertex algebras).

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