ON CERTAIN $W$-ALGEBRAS OF TYPE $W_k(sl_4, f)$

DRAŽEN ADAMOVIĆ, ANTUN MILAS, MICHAEL PENN

Abstract. We study several examples of simple $W$-algebras $W_k(sl_4, f)$ in connection to collapsing levels of affine vertex algebras and $N = 3$ conformal superalgebras.

1. Introduction

In their important work [FF1, FF2], B. Feigin and E. Frenkel showed that a vertex algebra $W^k(g)$ (called affine $W$-algebra) can be associated to a simple Lie algebra $g$ by means of a quantum reduction of the corresponding level $k$ universal affine vertex algebra $V^k(g)$. This construction made use of a principal nilpotent element of $g$. Further properties of $W^k(g)$ and its simple quotient $W_k(g)$ were investigated by E. Frenkel, V. Kac, M. Wakimoto [FKW] and T. Arakawa [Ar].

This construction can be generalized to any nilpotent element of $g$, which has been used by many authors culminating in the treatment of V. Kac, S. Roan, and M. Wakimoto ([KRW], [KW]) who proved important structure theorems.

It is generally unknown the structure of the simple vertex algebras $W_k(g, f)$ for any nilpotent element, e.g. a minimal generating set. In this short paper, our modest goal is to attempt to answer this question for a special class of $W$-algebras coming from the simple Lie algebra $sl_4$ at certain special levels and nilpotent elements. We should point out that affine $W$-algebras associated to $sl_4$ have already appeared in the literature [CL] but associated to different nilpotent elements.

Let us briefly outline the content of the paper. After setting up notation and structural results (e.g. Proposition 2.1), we first consider the level $k = -\frac{8}{3}$. This level is collapsing and we have an isomorphism with the affine vertex algebra of type $sl_2$ (cf. Theorem 3.1). We also have a related result for the level $k = -\frac{2}{3}$ (see Remark 3.2). Next we consider $k = -\frac{5}{2}$. This case is no longer collapsing. Instead, we get an isomorphism with a certain subalgebra of the rank two $\beta\gamma$-system, which is of type $\begin{array}{c} 1^3, 2^4 \end{array}$. This isomorphism was already stated in [CKLR]; here we only provide further details.

In Section 5, we consider the $W$-algebra coming from $W^k(spo(2|3), f_\theta)$, also known as the $N = 3$ superconformal algebra. Here we first outline the proof of Theorem 5.1 giving the structure of the even part of $W^k(spo(2|3), f_\theta)$, being of type $\begin{array}{c} 1^3, 2^3, 4^6, 5^3 \end{array}$. We then use this result to construct an isomorphism between $W_{-1/3}(spo(2|3), f_\theta)^{Z_2}$ and the simple affine VOA $L_{-2/3}(sl_2)$, cf. Corollary 5.1.

The $N = 3$ vertex superalgebra is often tensored with free fermions $F$. The internal structure of the even part of the corresponding algebra is given in Theorem 5.2. This is then used to prove the main result of Section 5 (cf. Theorem 5.3):

$$(W_{-1/3}(spo(2|3), f_\theta) \otimes F)^{Z_2} \cong W_{-7/3}(sl_4, f).$$
Acknowledgements: Results of the paper were presented by the third author at the conference "Representation Theory XVI", Dubrovnik, June 2019. We thank A. Linshaw for discussion regarding Theorems 4.1 and 4.2. Many computations in the paper are performed using Thielemans' OPE Mathematica package [T]

2. Affine \( W \)-algebras

We begin with a review of the necessary background in order to motivate our construction, which involves the Lie algebra \( \mathfrak{sl}_2 \) and a "short" nilpotent element.

2.1. \( W \)-algebra \( W^k(\mathfrak{g}, f) \). Let \( \mathfrak{g} \) be a simple, finite dimensional Lie algebra and \( h, f \in \mathfrak{g} \) be two parts of an \( \mathfrak{sl}_2 \)-triple, where \( f \) is nilpotent, and

\[
[h, f] = -f.
\]

Now decompose

\[
\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j
\]

by eigenvalues of \( \text{ad} \ h \), where \( a \in \mathfrak{g}_j \) if \([h, a] = ja\). Set

\[
\mathfrak{g}^f = \{ a \in \mathfrak{g} : [f, a] = 0 \}
\]

and observe that all eigenvalues of \( \text{ad} \ x \) on \( \mathfrak{g}^f \) are non-positive. Let \( A \) be the vector super-space of \( \mathfrak{g}_+ = \bigoplus_{j > 0} \mathfrak{g}_j \), where the parity has been reversed, and \( A^* \) be the dual space. Let \( \{b_\alpha | \alpha \in S_+\} \) and \( \{c_\alpha | \alpha \in S_+\} \) be bases for \( A \) and \( A^* \) respectively and set \( A_{ch} = A \oplus A^* \) and form the associated vertex algebra of charged free fermions \( \mathcal{F}(A_{ch}) \) where the nontrivial OPE is given by

\[
b_\alpha(z)c_\beta(z) \sim \frac{\delta_{\alpha, \beta}}{z - w}.
\]

Now let \( V^k(\mathfrak{g}) \) be the level \( k \neq -h^\vee \) universal affine vertex algebra associated to \( \mathfrak{g} \). We denote its simple quotient by \( L_k(\mathfrak{g}) \) (another standard notation is \( L_\mathfrak{g}(k\Lambda_0) \)).

Now we form

\[
\mathcal{C}(\mathfrak{g}, f, x) = V_k(\mathfrak{g}) \otimes \mathcal{F}(A_{ch}).
\]

and the following field of this vertex algebra

\[
d(z) = \sum_{\alpha \in S_+} u_\alpha(z) \otimes c_\alpha(z) - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in S_+} c_{\alpha, \beta}^\gamma \otimes b_{\gamma}(z)c_\alpha(z)c_\beta(z) + \sum_{\alpha \in S_+} (f|u_\alpha) \otimes c_\alpha(z),
\]

(2.1)

where the structure constants \( c_{\alpha, \beta}^\gamma \) are defined by \([u_\alpha, u_\beta] = \sum_{\gamma} c_{\alpha, \beta}^\gamma u_\gamma \) where \( \{u_\alpha\} \) forms a basis of \( \mathfrak{g} \). Define \( D = \text{Res}_z d(z) \). It is well known that \( D \) is a vertex algebra homomorphism and defines a \( \mathbb{Z} \)-graded homology complex \( (\mathcal{C}(\mathfrak{g}, f, x), D) \).

The associated homology is denoted \( W_k(\mathfrak{g}, f) \) and is called the quantum reduction of \( V_k(\mathfrak{g}) \) with respect to the nilpotent element \( f \in \mathfrak{g} \) (see [FF1, FF2, FBZ]).

Consider the following fields, for \( v \in \mathfrak{g} \),

\[
J^{(v)}(z) = v(z) \otimes 1 + \sum_{\alpha, \beta \in S_+} c_{\alpha, \beta}^\gamma(v) \otimes b_{\alpha}(z)c_\beta(z),
\]

where \( c_{\alpha, \beta}^\gamma(v) \) are structure constants defined by \([v, u_\beta] = \sum_{\alpha \in S} c_{\alpha, \beta}^\gamma(v)u_\alpha \) where \( \{u_\alpha | \alpha \in S\} \) is a basis of \( \mathfrak{g} \). By Theorem 4.1 of [KW] we know that for \( v \in \mathfrak{g}^f \)
We decompose \((2.2)\)

and \((2.4)\)

where we will set \(D\) and the root system \(\Delta = \Delta^+\). We say that a nilpotent element

A to check that where this process ends in finitely many steps, where \(D\)

We use non-degenerate bilinear form to identify \(-g\)

Also consider the centralizer of \(f\)

In order to find the homology class for \(\mathbb{V}\)

Remark 1. In order to find the homology class for \(v\) we follow the strategy outlined in [BT]. This involves splitting \(d(z) = d_1(z) + d_2(z)\), where

\[
d_1(z) = \sum_{\alpha \in S_+} u_\alpha(z) \otimes c_\alpha(z) - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in S_+} c^\gamma_{\alpha,\beta} \otimes \gamma b_\gamma(z) c_\alpha(z)c_\beta(z)z\]

and

\[
d_2(z) = \sum_{\alpha \in S_+} (f|u_\alpha) \otimes c_\alpha(z),
\]

where we will set \(D_j = \text{Res}_zd_j(z)\). Now, for \(v \in g\) we define

\[
W_v(z) = \sum_{j=0}^m (-1)^j W_j(v) (z),
\]

where \(W_0^v(z) = J^v(z)\) and for \(j \geq 1\), \(W_j^v(z)\) is chosen so that

\[
D_2(W_j^v(z)) = D_1(W_j^v(z)),
\]

where this process ends in finitely many steps, where \(D_2(W_0^v(z)) = 0\). It is easy to check that \(D_1(J^v(z)) = 0\) for \(v \in g\) and thus \(D(W^v(z)) = 0\).

2.2. A \(\mathbb{V}\)-algebra associated to \(\mathfrak{sl}_2\) and a short nilpotent element. We now move to the example of a quantum reduction that will be of interest in our setting. Denote the positive roots of \(\mathfrak{sl}_4\) by

\[
\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}
\]

and the root system \(\Delta = \Delta_+ \cup \Delta_-\), where \(\Delta_- = -\Delta_+\). We may realize \(g = \mathfrak{sl}_4\) by

\[
g = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} x_\alpha.
\]

We use non-degenerate bilinear form to identify \(\mathfrak{h}\) with \(\mathfrak{h}^*\).

Definition 2.1. We say that a nilpotent element \(f \in g\) is called short if the ad\(f\) eigenvalues are \(-1, 0, 1\), namely we have decomposition \(g = g_{-1} \oplus g_0 \oplus g_1\).

Now consider the short nipotent element \(f = x_{-\alpha_1 - \alpha_2} + x_{-\alpha_2 - \alpha_3}\) and its corresponding \(\mathfrak{sl}_2\) triple completed by \(e = x_{\alpha_1 + \alpha_2} + x_{\alpha_2 + \alpha_3}\) and \(h = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \alpha_3)\). We decompose \(g\) into \(ad\ h\) eigenspaces

\[
g_{-1} = \text{span} \{x_{-\alpha_2}, x_{-\alpha_1 - \alpha_2}, x_{-\alpha_2 - \alpha_3}, x_{-\alpha_1 - \alpha_2 - \alpha_3}\}
\]

\[
g_0 = \text{span} \{x_{\alpha_1}, x_{\alpha_2}, x_{-\alpha_1}, x_{-\alpha_2}, x_{\alpha_1 \alpha_2}, x_{\alpha_1 + \alpha_2}, x_{\alpha_2 + \alpha_3}\}
\]

\[
g_1 = \text{span} \{x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_1 + \alpha_2}, x_{\alpha_2 + \alpha_3}, x_{\alpha_1 + \alpha_2 + \alpha_3}\}
\]

Also consider the centralizer of \(f\)

\[
g^f = \text{span} \{\alpha_1 + \alpha_3, x_{\alpha_1 + \alpha_3}, x_{-\alpha_2 - \alpha_3}, x_{-\alpha_1 + \alpha_2}, x_{-\alpha_1 + \alpha_2}, x_{-\alpha_1 - \alpha_2 - \alpha_3}, x_{-\alpha_1 - \alpha_2 - \alpha_3}\}
\]

and its decomposition into \(ad\ h\) eigenspaces

\[
g^f_{(-1)} = \text{span} \{x_{-\alpha_2}, x_{\alpha_1 + \alpha_2}, x_{-\alpha_2 - \alpha_3}, x_{-\alpha_1 - \alpha_2 - \alpha_3}\}
\]

\[
g^f_{(0)} = \text{span} \{\alpha_1 + \alpha_3, x_{\alpha_1}, x_{\alpha_3}, x_{-\alpha_1} + x_{-\alpha_3}\}.
\]
Set $S_+ = \{x_{\alpha_2}, x_{\alpha_1+\alpha_2}, x_{\alpha_2\alpha_1}, x_{\alpha_1+\alpha_2+\alpha_3}\}$ and the corresponding vector spaces

$$A = \text{span}\ \{b_{\alpha_2}, b_{\alpha_1+\alpha_2}, b_{\alpha_1+\alpha_2+\alpha_3}\}$$
(2.5)
$$A^* = \text{span}\ \{c_{\alpha_2}, c_{\alpha_1+\alpha_2}, c_{\alpha_1+\alpha_2+\alpha_3}\}$$
$$A_{\text{ch}} = A \oplus A^*,$$

and construct the vertex algebra of free charged fermions $\mathcal{F}(A_{\text{ch}})$ as described above. In this setting we have

$$d_1(z) = c_{\alpha_1+\alpha_2} + c_{\alpha_2+\alpha_3}$$

and

$$d_2(z) = \circ x_{\alpha_2} c_{\alpha_2} + \circ x_{\alpha_1+\alpha_2} c_{\alpha_1+\alpha_2} + \circ x_{\alpha_2+\alpha_3} c_{\alpha_2+\alpha_3} + \circ x_{\alpha_1+\alpha_2+\alpha_3}.\]$$

An application of Theorem 4.1 of [KW] gives (see also [AMo]):

**Proposition 2.1.** The vertex algebra $\mathcal{W}^k(\mathfrak{sl}_4, f)$ is strongly generated by the homology classes of the fields

$$J^{(\alpha_1+\alpha_3)}(z) = \alpha_1(z) + \alpha_3(z) + 2\circ b_{\alpha_1+\alpha_2} c_{\alpha_1+\alpha_2} + 2\circ b_{\alpha_1+\alpha_2+\alpha_3} c_{\alpha_1+\alpha_2+\alpha_3}$$
$$J^{(x_{\alpha_1}+x_{\alpha_2})}(z) = x_{\alpha_1}(z) + x_{\alpha_2}(z) + \circ b_{\alpha_1+\alpha_2+\alpha_3} c_{\alpha_1+\alpha_2+\alpha_3} + \circ b_{\alpha_2+\alpha_3} c_{\alpha_2+\alpha_3} - \circ b_{\alpha_1+\alpha_2} c_{\alpha_2+\alpha_3} - \circ b_{\alpha_1+\alpha_2+\alpha_3} c_{\alpha_1+\alpha_2+\alpha_3}$$
$$J^{(x_{-\alpha_1}+x_{-\alpha_3})}(z) = x_{-\alpha_1}(z) + x_{-\alpha_3}(z) + \circ b_{\alpha_2} c_{\alpha_2+\alpha_3} + \circ b_{\alpha_1+\alpha_2} c_{\alpha_1+\alpha_2+\alpha_3} - \circ b_{\alpha_2} c_{\alpha_2+\alpha_3} - \circ b_{\alpha_1+\alpha_2} c_{\alpha_1+\alpha_2+\alpha_3}$$
$$J^{(x_{-\alpha_2})}(z) = x_{-\alpha_2}(z)$$
$$J^{(x_{-\alpha_1}-x_{-\alpha_2})}(z) = x_{-\alpha_1}-x_{-\alpha_2}(z)$$
$$J^{(x_{-\alpha_2}-x_{-\alpha_3})}(z) = x_{-\alpha_2}-x_{-\alpha_3}(z)$$
$$J^{(x_{-\alpha_1}-x_{-\alpha_2}-x_{-\alpha_3})}(z) = x_{-\alpha_1}-x_{-\alpha_2}-x_{-\alpha_3}(z)$$

of conformal weights $1, 1, 1, 2, 2, 2$ respectively.

Let us denote by $\mathcal{W}_k(\mathfrak{sl}_4, f)$ the simple quotient of $\mathcal{W}^k(\mathfrak{sl}_4, f)$. 

$$\text{span}\ \{x_{\alpha_2}, x_{\alpha_1+\alpha_2}, x_{\alpha_2\alpha_1}, x_{\alpha_1+\alpha_2+\alpha_3}\}$$
Following the iterative process described above (2.2-2.3) we calculate the following generators:

\[(2.6)\]
\[
W^{(x_{a_1}+x_{a_3})}(z) = J^{(x_{a_1}+x_{a_3})}(z)
\]
\[
W^{(x_{-a_1}+x_{a_3})}(z) = J^{(x_{-a_1}+x_{a_3})}(z)
\]
\[
W^{(\alpha_1+\alpha_3)}(z) = J^{(\alpha_1+\alpha_3)}(z)
\]
\[
W^{(x_{-a_1}+2)}(z) = J^{(x_{-a_1}+2)}(z) - \frac{1}{4}\partial J^{(\alpha_1)} J^{(\alpha_1)} - \frac{1}{2} \partial J^{(\alpha_2)} J^{(\alpha_2)} - \frac{1}{4} \partial J^{(\alpha_3)} J^{(\alpha_3)}
\]
\[
W^{(x_{-a_2}+3)}(z) = J^{(x_{-a_2}+3)}(z) - \frac{1}{4}\partial J^{(\alpha_2)} J^{(\alpha_2)} - \frac{1}{2} \partial J^{(\alpha_3)} J^{(\alpha_3)} - \frac{1}{4} \partial J^{(\alpha_3)} J^{(\alpha_3)}
\]
\[
W^{(x_{-a_1}+x_{a_2})}(z) = J^{(x_{-a_1}+x_{a_2})}(z) - \frac{1}{4}\partial J^{(\alpha_1)} J^{(\alpha_1)} - \frac{1}{2} \partial J^{(\alpha_2)} J^{(\alpha_2)} - \frac{1}{4} \partial J^{(\alpha_3)} J^{(\alpha_3)} + \frac{1}{2} \partial J^{(\alpha_3)} J^{(\alpha_3)}
\]
\[
W^{(x_{-a_2})}(z) = J^{(x_{-a_2})} + \frac{1}{2} \partial J^{(\alpha_2)} J^{(\alpha_2)} - \frac{1}{2} \partial J^{(\alpha_3)} J^{(\alpha_3)} - \frac{1}{4} \partial J^{(\alpha_3)} J^{(\alpha_3)}
\]

where the Virasoro field is given by

\[(2.7)\]
\[
L(z) = -\frac{1}{k+4} \left(W^{(x_{-a_1}+2)}(z) + W^{(x_{-a_2}+3)}(z) - W^{(x_{a_1}+x_{a_3})} W^{(x_{-a_1}+x_{a_3})} + \frac{1}{8} W^{(\alpha_1+\alpha_3)} W^{(\alpha_1+\alpha_3)} - \frac{k+1}{2} \partial W^{(\alpha_1+\alpha_3)} \right)
\]

and has central charge

\[(2.8)\]
\[
c_V = -\frac{12k^2 + 41k + 32}{k+4}.
\]

We make the following change of variables on the remaining fields for notational convenience into the following primary fields of weight 1, 1, 2, 2, 2 respectively.

\[
J^+(z) = W^{(x_{a_1}+x_{a_3})}(z)
\]
\[
J^-(z) = W^{(x_{-a_1}+x_{a_3})}
\]
\[
J^0(z) = W^{(\alpha_1+\alpha_3)}(z)
\]
\[
H(z) = W^{(x_{-a_1}+2)}(z) - W^{(x_{-a_2}+3)}(z) - \frac{1}{2} \partial W^{(\alpha_1+\alpha_3)}(z)
\]
\[
E(z) = W^{(x_{-a_2})} + \frac{1}{4} \partial W^{(\alpha_1+\alpha_3)} W^{(x_{a_1}+x_{a_3})} - \frac{1}{2} \partial W^{(x_{a_1}+x_{a_3})}(z)
\]
\[
F(z) = W^{(x_{-a_1}+x_{a_2})}(z) + \frac{1}{4} \partial W^{(\alpha_1+\alpha_3)} W^{(x_{-a_1}+x_{a_3})} - \frac{k+2}{2} \partial W^{(x_{-a_1}+x_{a_3})}(z)
\]
The fields $J^+, J^-$, and $J^0$ generate a sub-VOA that is isomorphic to $V^{2k+4}(\mathfrak{sl}_2)$, i.e., the level $2k + 4$ universal VOA associated to $\mathfrak{sl}_2$. The remaining nontrivial OPE are given by \(^1\):

\[
\begin{align*}
J^+ H & \sim \frac{2E}{z-w}, & J^+ F & \sim \frac{-H}{z-w}, \\
J^- H & \sim \frac{-2F}{z-w}, & J^- E & \sim \frac{H}{z-w}, \\
J^0 F & \sim \frac{-2F}{z-w}, & J^0 E & \sim \frac{2E}{z-w}, \\
HF & \sim \frac{-\frac{1}{2}(24 + 25k + 6k^2)J^-}{(z-w)^3} + \frac{-\frac{1}{4}(8 + 3k)^2 J^0 J^-}{(z-w)^2} - \frac{1}{2}(16 + 16k + 3k^2)\partial J^- \\
& \quad \quad + \frac{(4 + k)^2 J^- L^+_o - \frac{1}{4} J^0 J^-}{z-w} + \frac{-\frac{1}{2}(4 + k)^2 J_0 \partial J^-}{z-w} + \frac{\frac{1}{2}(2 - k)^2 J_0 \partial J^-}{z-w} + \frac{\frac{1}{2}(6 + 5k + k^2) \partial^2 J^-}{z-w}, \\
HE & \sim \frac{\frac{1}{2}(24 + 25 + 2k^2)J^+}{(z-w)^3} + \frac{-\frac{1}{4}(8 + 3k)^2 J^0 J^+}{(z-w)^2} + \frac{1}{2}(16 + 14k + 3k^2)\partial J^+ \\
& \quad \quad + \frac{\frac{1}{4}(4 + k)^2 J^+ L^+_o + \frac{1}{4} J^0 J^+}{z-w} - \frac{1}{2}(4 + k)^2 J_0 \partial J^+}{z-w} + \frac{-\frac{1}{4}(6 + k)^2 J_0 \partial J^+}{z-w} + \frac{\frac{1}{2}(8 + 5k + k^2) \partial^2 J^+}{z-w}, \\
HH & \sim \frac{-\frac{1}{2}(48 + 74k + 37k^2 + 6k^3)}{(z-w)^4} + \frac{2(2 + k)(4 + k)L - (8 + 4k)^2 J^- \partial J^-}{(z-w)^2} - \frac{k^2 J_0 J^-}{(z-w)^2} + \frac{(4 + 2k) \partial^2 J^-}{z-w}, \\
EF & \sim \frac{-\frac{1}{2}(48 + 74k + 37k^2 + 6k^3)}{(z-w)^4} + \frac{-\frac{1}{4}(24 + 25k + 6k^2)J^0}{(z-w)^3} + \frac{(2 + k)(4 + k) \partial L - (4 + 2k)^2 J^- \partial J^-}{(z-w)^2} - \frac{1}{2}(8 + 5k + 3k^2) J^- \partial J^0} \\
& \quad \quad + \frac{\frac{1}{4}(4 + k)^2 \partial J^- J^-}{z-w} - \frac{1}{2}(3 + 2k)^2 \partial J^0 J^-}{z-w} + \frac{-\frac{1}{4}(2 + 3k + k^2) \partial^2 J^0}{z-w}, \\
EE & \sim \frac{\frac{1}{4}(8 + 3k)^2 J^- J^+}{(z-w)^2}, \\
FF & \sim \frac{\frac{1}{4}(8 + 3k)^2 J^- J^-}{(z-w)^2}.
\end{align*}
\]

\(^1\)For brevity we write ab instead of a(z)b(w).
3. The collapsing level $k = -\frac{8}{3}$

We now examine a special case of $W_k(\mathfrak{sl}_4, f)$. Here we set $k = -\frac{8}{3}$ and to motivate the upcoming result consider the following elements

\begin{align}
\hat{A} &= \frac{1}{48}(-9zHH^o - 24zXY^o - 4\partial H) \\
\hat{B} &= \frac{1}{16}(-3zHH^o - 8zXY^o + 4\partial H) \\
\hat{P} &= \frac{1}{12}(-3zHY^o - 4\partial Y) \\
\hat{Q} &= \frac{1}{12}(zHX^o - 4\partial X).
\end{align}

and consider the map (not a vertex algebra homomorphism) from $W_{-\frac{8}{3}}(\mathfrak{sl}_4, f)$ to its subalgebra generated by $X, Y, H$ that sends each field to its “hatted” version, where $\hat{X} = X$, $\hat{Y} = Y$, and $\hat{H} = H$. The OPE of these fields only differ from the OPE for the fields $A, B, P$, and $Q$ in the following ways

\begin{align}
\hat{A}P - \hat{A}\hat{P} &\sim \frac{1}{72(z-w)}(-9zHHY^o - 48zH\partial Y^o - 36zXYY^o + 66z\partial HY^o - 4\partial^2 Y) \\
\hat{A}Q - \hat{A}\hat{Q} &\sim \frac{1}{72(z-w)}(9zHHX^o - 48zH\partial X^o - 36zXXY^o - 6\partial HX^o + 40\partial^2 X) \\
\hat{B}P - \hat{B}\hat{P} &\sim \frac{1}{72(z-w)}(9zHHY^o + 48zH\partial Y^o + 36zXYY^o - 66z\partial HY^o - 4\partial^2 Y) \\
\hat{B}Q - \hat{B}\hat{Q} &\sim \frac{1}{72(z-w)}(-9zHHX^o + 48zH\partial X^o + 36zXXY^o - 6\partial HX^o - 40\partial^2 X) \\
\hat{P}Q - \hat{P}\hat{Q} &\sim \frac{1}{72(z-w)}(9zHHH^o - 24zX\partial Y^o + 36zXXH^o + 24z\partial XY^o + 4\partial^2 H).
\end{align}

Fortunately, a routine calculation checks that for the level $2\left(-\frac{8}{3} + 2\right) = -\frac{4}{3}$ universal affine vertex algebra $V^{-4/3}(\mathfrak{sl}_2)$ these fields are in the maximal ideal used to form the corresponding simple VOA, $L_{-4/3}(\mathfrak{sl}_2)$. Therefore we obtained the following result.

**Theorem 3.1.** We have a surjective vertex algebra homomorphism

\[ \varphi : W_{-\frac{8}{3}}(\mathfrak{sl}_4, f) \to L_{-4/3}(\mathfrak{sl}_2) \]

In particular,

\[ W_{-\frac{8}{3}}(\mathfrak{sl}_4, f)/I \cong L_{-4/3}(\mathfrak{sl}_2) \]

where

\[ I = J + \langle A - \hat{A}, B - \hat{B}, P - \hat{P}, Q - \hat{Q} \rangle \]

and $J$ is the maximal ideal of the copy of $V^{-4/3}(\mathfrak{sl}_2)$ in $W_{-\frac{8}{3}}(\mathfrak{sl}_4, f)$ such that

\[ L_{-4/3}(\mathfrak{sl}_2) = V^{-4/3}(\mathfrak{sl}_2)/J. \]

3.1. A different proof of Proposition 3.1.

**Remark 3.1.** One can give an alternative proof of Proposition 3.1 based on the following facts:
3.2. Collapsing level: Using the analysis in Remark 3.1, we can easily prove that

\[ \text{Remark 3.2.} \]

Consider the field \( h \) of conformal weight 4 and the \( sl_2 \) system, without a proof in Remark 5.3 of [CKLR]. Here we provide additional details.

(i) Since VOA \( L_{-8/3}(sl_4) \) is admissible, it is known that the maximal ideal \( J^{-8/3} \) in the universal affine voa \( V^{-8/3}(sl_4) \) is generated by a singular vector \( \Omega_{-8/3} \) of \( sl_4 \)-weight \( \omega_1 + \omega_3 = \theta \) (=highest root) and conformal weight 3. By using exact functor \( H_{\tilde{f}}^{\tilde{S}} \) from [ArM] (all details are covered in [ArM]), we get that \( H_{\tilde{f}}^{\tilde{S}}(J^{-8/3}) \) is a non-trivial ideal in \( V^{k}(sl_4,f) \).

(ii) Recall that \( W^{k}(sl_4,f) \) is strongly generated by the generators of \( V^{k}(sl_4,f) \), the Virasoro vector \( \omega \) and the three dimensional subspace \( U_2 \subset W^{k}(sl_4,f) \) such that

\[ V^{k}(sl_2).U_2 \cong V^{k}(2\omega). \]

(iii) The lowest graded component of \( H_{\tilde{f}}^{\tilde{S}}(J^{-8/3}) \) has conformal weight 2 (= conformal weight of \( \Omega_{-8/3} \) minus one, which is 3–1 = 2) and \( sl_2 \)-weight \( \theta^4 \). Because there are no singular vectors of conformal weight less than three in \( V_{-4/3}(sl_2) \), we get

\[ U_2 \subset H_{\tilde{f}}^{\tilde{S}}(J^{-8/3}). \]

(iv) Therefore \( W^{-8/3}(sl_4,f)/H_{\tilde{f}}^{\tilde{S}}(J^{-8/3}) \) is generated by \( V^{-8/3}(sl_2) \) and possible the Virasoro vector \( \omega \). Since

\[ x(n)(\omega - \omega_{sug}) = L(n)(\omega - \omega_{sug}) = 0 \quad (n \geq 0, \ x \in \mathfrak{s}l_2) \]

we conclude \( \omega - \omega_{sug} \) also belongs to this quotient. So \( \omega = \omega_{sug} \).

(v) Therefore \( \Omega_{-3/2}(sl_4,f)/H_{\tilde{f}}^{\tilde{S}}(J^{-8/3}) \) collapses to an affine VOA of level \( -4/3 \). But using results from [ArM] again, we get that \( H_{\tilde{f}}^{\tilde{S}}(L_{-8/3}(sl_4)) \) is a simple VOA, and therefore we have that \( W_{-3/2}(sl_4,f) \cong L_{1}(sl_4) \).

3.2. Collapsing level: \( k = -3/2 \).

**Remark 3.2.** Using the analysis in Remark 3.1, we can easily prove that

\[ W_{-3/2}(sl_4,f) \cong L_{1}(sl_4). \]

We consider the singular vector \( \Omega_{-3/2} \) in the admissible affine vertex algebra \( V^{-3/2}(sl_4) \) of conformal weight 4 and the \( sl_4 \)-weight \( 2\omega_1 \).

4. The level \( k = -5/2 \) and a certain Heisenberg coset

In this section we prove that, if \( k = -5/2 \), our algebra is isomorphic to a coset of a Heisenberg vertex algebra inside of a rank two \( \beta \gamma \) system. This result was originally state without a proof in Remark 5.3 of [CKLR]. Here we provide additional details.

We recall the results of the previous construction of this algebra, calculate the OPE of the generating fields, and describe the relation with our algebra.

Consider the rank 2 \( \beta \gamma \) system, \( S(2) \), generated by even, weight 1/2 fields \( \beta_1, \beta_2, \gamma_1, \gamma_2 \) subject to the non-trivial OPE

\[ \beta_i \gamma_j \sim \frac{\delta_{i,j}}{z-w}. \]

Consider the field \( h = z \beta_1 \gamma_1 + z \beta_2 \gamma_2 \), which generates a rank 1 Heisenberg subalgebra of \( S(2) \), which we denote by \( \mathcal{H} \). Finally consider the coset \( C(2) = \text{Com}(\mathcal{H},S(2)) \). We now recall a theorem of [CKLR].
Theorem 4.1 ([CKLR] Theorem 5.3). C(2) is simple and of type $W(1, 1, 1, 2, 2, 2)$. In fact, it is the simple quotient of an algebra of type $W(1, 1, 1, 2, 2, 2)$ where the Virasoro field in weight 2 coincides with the Sugawara field. Moreover, explicit primary generators are given by

\[(4.1)\]

$$
x_{1,2} = -\gamma_2\gamma_1, \\
x_{2,1} = -\gamma_2\gamma_1, \\
h_1 = -\gamma_1\gamma_1 + \gamma_2\gamma_2,
$$

\[P = \frac{\partial}{\partial z}\gamma_2 - \frac{\partial}{\partial w}\gamma_1 + \frac{1}{3}\gamma_1\gamma_1\gamma_1 + \frac{2}{3}\gamma_1\gamma_2\gamma_2, \]

\[Q = \frac{\partial}{\partial z}\gamma_2 - \frac{\partial}{\partial w}\gamma_1 + \frac{1}{3}\gamma_1\gamma_1\gamma_1 + \frac{2}{3}\gamma_1\gamma_2\gamma_2, \]

\[R = \frac{\partial}{\partial z}\gamma_1 - \frac{\partial}{\partial w}\gamma_1 + 2\delta\gamma_1\gamma_1 - 2\delta\gamma_2\gamma_2 - 2\delta(\partial\gamma_1)\gamma_1 - 2\delta(\partial\gamma_2)\gamma_2, \]

where $x_{1,2}, x_{2,1}, h_1$ generate a subalgebra isomorphic to $V_{-1}(sl_2)$ and the Virasoro field is given by $L = \frac{1}{3}x_{1,2}x_{2,1} + \frac{1}{4}h_1h_1 - \frac{1}{2}\partial h_1$.

The remaining nontrivial OPE of the generators (4.1) are given by:

$$
h_1P \sim \frac{3}{2}x_{1,2} + \frac{2P}{z - w}, \\
h_1Q \sim \frac{2}{3}x_{2,1} + \frac{-2Q}{z - w}, \\
x_{1,2}P \sim \frac{1}{3}x_{1,2}x_{1,2} - \frac{1}{2}h_1, \\
x_{1,2}Q \sim \frac{1}{2}h_1 + \frac{2}{3}L + \frac{1}{2}R + \frac{1}{3}x_{1,2}x_{2,1} - \frac{1}{2}\partial h_1.
$$
\[ x_{1,2} R \sim \frac{-4P - \frac{2}{3}h_1 x_{1,2} \dot{\circ} + \frac{2}{3} \partial x_{1,2}}{z - w} \]
\[ x_{2,1} P \sim \frac{\frac{1}{3}h_1 + \frac{2}{3}L - \frac{1}{2}R - \frac{5}{3}x_{1,2}x_{2,1} + \frac{1}{2} \partial h_1 \dot{\circ}}{z - w} \]
\[ x_{2,1} Q \sim \frac{- \frac{1}{3}x_{2,1} x_{1,2} \dot{\circ}}{z - w} \]
\[ x_{2,1} R \sim \frac{4Q + \frac{2}{3}h_1 x_{2,1} \dot{\circ} + \frac{2}{3} \partial x_{2,1}}{z - w} \]
\[ PP \sim \frac{\frac{2}{3}x_{1,2} x_{2,1} \dot{\circ} + \frac{2}{3} \partial x_{1,2} \partial x_{1,2} \dot{\circ}}{(z - w)^2} + \frac{\frac{2}{3} \partial x_{1,2} \partial x_{1,2} \dot{\circ}}{(z - w)} \]
\[ PQ \sim \frac{\frac{-8}{3} \partial h_1 + \frac{8}{3}h_1 - \frac{R + \frac{13}{9}x_{1,2} x_{2,1} \dot{\circ} + \frac{2}{3} \partial h_1 \dot{\circ} - \frac{5}{18} \partial h_1}{(z - w)^2}}{z - w} \]
\[ + \frac{\frac{1}{3} h_1 R \dot{\circ} + \frac{2}{3} x_{2,1} P \dot{\circ} + \frac{h_1 x_{1,2} x_{2,1} \dot{\circ} + \frac{5}{18} h_1 h_1 \dot{\circ}}{z - w}}{z - w} \]
\[ PR \sim \frac{-4x_{1,2} \partial x_{1,2} \dot{\circ} + \frac{2}{3} P + \frac{19}{9} h_1 x_{1,2} \dot{\circ} - \frac{28}{3} \partial x_{1,2}}{(z - w)^3} + \frac{\frac{2}{3} \partial P - \frac{4}{3} x_{1,2} L \dot{\circ} + \frac{1}{3} h_1 x_{1,2} R \dot{\circ} - \frac{17}{9} (\partial h_1) x_{1,2} \dot{\circ}}{(z - w)} \]
\[ QQ \sim \frac{\frac{2}{3} x_{2,1} x_{2,1} \dot{\circ} + \frac{2}{3} \partial x_{2,1} \dot{\circ}}{(z - w)^2} + \frac{\frac{2}{3} \partial x_{2,1} \dot{\circ}}{(z - w)} \]
\[ QR \sim \frac{\frac{-4 x_{2,1} \partial x_{2,1} \dot{\circ}}{(z - w)^3} + \frac{4}{3} Q + \frac{19}{9} h_1 x_{2,1} \dot{\circ} + \frac{28}{3} \partial x_{2,1}}{(z - w)^2} + \frac{\frac{3}{2} \partial Q + \frac{4}{3} x_{2,1} L \dot{\circ} - \frac{1}{3} h_1 x_{1,2} R \dot{\circ} - \frac{17}{9} (\partial h_1) x_{2,1} \dot{\circ}}{z - w} \]
\[ RR \sim \frac{-8}{(z - w)^4} + \frac{- \frac{184}{3} L + \frac{82}{9} h_1 h_1 \dot{\circ} + \frac{289}{3} x_{1,2} x_{2,1} \dot{\circ} - \frac{128}{9} \partial h_1}{(z - w)^2} \]
\[ + \frac{- \frac{184}{9} \partial L + \frac{58}{3} h_1 \partial h_1 + \frac{104}{3} (\partial x_{1,2}) x_{2,1} \dot{\circ} + \frac{104}{3} x_{1,2} \partial x_{2,1} \dot{\circ} - \frac{52}{3} \partial^2 h_1}{z - w} \]

We now return to our algebra, \( W_{-5/2}(\mathfrak{sl}_4, f) \). Consider the following fields:
\[
\hat{P} = 2E - \frac{1}{6} J^0 J^+ \dot{\circ} + \frac{1}{6} \partial J^+ \\
\hat{Q} = 2F - \frac{1}{6} J^0 J^- \dot{\circ} - \frac{1}{6} \partial J^- \\
\hat{R} = -4H + \frac{4}{3} L - \frac{4}{3} J^+ J^- \dot{\circ} - \frac{1}{3} J^0 J^0 \dot{\circ} + \frac{2}{3} \partial J^0,
\]
as well as
\[
T = L - \frac{4}{3} J^+ J^- \dot{\circ} - \frac{1}{4} J^0 J^0 \dot{\circ} + \frac{1}{2} \partial J^0.
\]

It is straightforward to check that, if we identify \( x_{1,2} \) with \( J^+ \), \( x_{2,1} \) with \( J^- \), and \( h_1 \) with \( J^0 \) the OPE algebra of the algebra \( W_{-5/2}(\mathfrak{sl}_4, f) \) agrees with the OPE
algebra of $C(2)$ up to fields that are in the ideal containing $T$. Explicitly, we have the following [CKLR]:

**Theorem 4.2.** We have

$$W_{-5/2}(sl_4, f) \cong C(2).$$

That is, we have a surjective vertex operator algebra homomorphism

$$\Phi : W_{-5/2}(sl_4, f) \to C(2)$$

whose mapping is described above and kernel contains the field $T$.

**Proof.** It is clear the $\Phi(T) = 0$ and thus $T \in \ker \Phi$. As stated above, it is a straightforward albeit involved calculation, to check that the OPE algebra of the algebra $W_{-5/2}(sl_4, f)$ agrees with the OPE algebra of $C(2)$ up to fields that are in the ideal containing $T$, and thus $\ker \Phi$. The differences in the OPE algebra are given by the following

$$\Phi(\hat{P}_1 \hat{Q}) = \phi(\hat{P})_1 \phi(\hat{Q}) + \frac{29}{9} f(T)$$

$$\Phi(\hat{P}_0 \hat{Q}) = \phi(\hat{P})_0 \phi(\hat{Q}) - \frac{31}{18} \phi(\partial T) + \frac{29}{9} \phi(\hat{\sigma} J^0 T^z) + \frac{1}{3} \phi(T H)$$

$$\Phi(\hat{P}_0 \hat{R}) = \phi(\hat{P})_0 \phi(\hat{R}) - \frac{76}{9} \phi(\hat{\sigma} T J^+ \hat{z}) - 4 \phi(T_0 E)$$

$$\Phi(\hat{Q}_0 \hat{R}) = \phi(\hat{Q})_0 \phi(\hat{R}) + \frac{76}{9} \phi(\hat{\sigma} T J^- \hat{z}) - 4 \phi(T_0 F)$$

$$\Phi(\hat{R}_0 \hat{R}) = \phi(\hat{R})_0 \phi(\hat{R}) + \frac{92}{9} \phi(T)$$

$$\Phi(\hat{R}_0 \hat{R}) = \phi(\hat{R})_0 \phi(\hat{R}).$$

$\Box$

5. **Case $k = -7/3$**

In this section we focus on the connection of $W_{-7/3}(sl_4, f)$ with the $N = 3$ superconformal algebra realized by the quantum Drinfeld-Sokolov reduction

$$W_{-1/3}(spo(2|3), f_{\theta})$$

as described in [KW], where $f_{\theta}$ is a minimal nilpotent, that is a lowest root vector of $\mathfrak{g}$. Our main result of this section will be the construction of the identification

$$W_{-7/3}(sl_4, f) \cong (W_{-1/3}(spo(2|3), f_{\theta}) \otimes \mathcal{F})^{\mathbb{Z}_2},$$

where $\mathcal{F}$ is a rank one neutral free fermion vertex operator (super) algebra. Building up to this result, we find the $\mathbb{Z}_2$ orbifold of both $W_{k}(spo(2|3), f_{\theta})$ and

$$W_{k}(spo(2|3), f_{\theta}) \otimes \mathcal{F}$$

for generic $k$. 
Remark 2. By [A, Theorem 8.2] the minimal affine $W$-algebra $W_{-1/3}(\text{spo}(2,3), f_0)$ of central charge $\bar{c} = -3/2$ is realized as a subalgebra of $SW(1) \otimes V_L$, the tensor product of the supertriplet vertex algebra $SW(1)$ [AM] with a lattice vertex algebra of central charge $c = 1$. It is also proved in [A] and [AKMPP] that

$$W_{-1/3}(\text{spo}(2,3), f_0) = V_{-2/3}(\mathfrak{sl}_2) \bigoplus V_{-2/3}(2\omega_1).$$

In particular, the algebra $W_{-1/3}(\text{spo}(2,3), f_0)$ is a superconformal extension of $V_{-2/3}(\mathfrak{sl}_2)$ and it is generated by odd vectors of conformal weight $3/2$: $G^\pm, G^0$ and by generators of $V_{-2/3}(\mathfrak{sl}_2)$. In what follows, we present a different proof of the structure of orbifold of $W_{-1/3}(\text{spo}(2,3), f_0)$.

5.1. The orbifold $W^k(\text{spo}(2|3), f_0)^{Z_2}$.

In [KW] the algebra $W^k(\text{spo}(2|3), f_0)$ is constructed and the OPEs are given. As such, we will only recall that this algebra is generated by three fields of weight $1$, $j^0$ and $j^\pm$ which generated a sub-VOA isomorphic to $V^{-2-4k}(\mathfrak{sl}_2)$, a Virasoro field, $L$, of central charge $-7/2 - 6k$, and three odd weight $3/2$ fields $G^0$ and $G^\pm$. It is clear that $\langle \Psi \rangle \cong Z_2 \subset \text{Aut}(W^k(\text{spo}(2|3), f_0))$ where

$$\Psi(G^0) = -G^0 \text{ and } \Psi(G^\pm) = -G^\pm,$$

and the action on all other generators is trivial. By the classical invariant theory of $Z_2 \cong \mathcal{O}(1)$ it is clear that in addition to the generators of the parent algebra that are fixed, $j^0, j^\pm$, and $L$, the orbifold is generated by the fields

$$W^{i,j}(a, b) = \partial^a G^i \partial^b G^j,$$

of weight $3 + a + b$, where $i, j \in \{0, +, -\}$ $i \leq j$ with $0 < + < -$, for $a, b \geq 0$. Following the reduction strategy in [CFPS], using the $\partial$ operator, we make an initial reduction of this set to

$$W^{i,j}(0, 2a + 1) \text{ and } W^{i,j}(0, b),$$

for $i, j \in \{0, +, -\}$ with $i < j$ and $a, b \geq 0$. Further, we know that the classical odd relations will give way to expressions of the form

$$W^{i_0,j_0}(a_0, b_0)W^{i_1,j_1}(a_1, b_1)^a + \omega W^{i_0,j_0}(a_0, b_0)W^{i_1,j_1}(a_1, b_1)$$

with $(i_0, j_0, a_0, b_0) < (i_1, j_1, a_1, b_1)$ ordered lexicographically, with quantum corrections that will provide a meaningful reduction of the generating set (5.4). Using the standard methods of decoupling relations and boot-strapping operator we obtain the following result (full details will appear in a separate publication [P]):

**Theorem 5.1.** For $k \neq -\frac{1}{3}(5\pm 3i\sqrt{7})$, the orbifold $W^k(\text{spo}(2|3), f_0)^{Z_2}$ is minimally generated by the fields $j^0, j^\pm, L, W^{0,0}(0,1), W^{0,\pm}(0,0), W^{\pm,0}(0,1), W^{0,\pm}(0,2), W^{+, -}(0,0), W^{+, -}(0,1), \text{ and } W^{+, -}(0,2)$ and is of type $(1^3, 2, 3^2, 4^6, 5^3)$.

Using results on conformal embeddings from [AKMPP], we obtain that, for $k = -\frac{1}{3}$, the maximal ideal $I$, such that

$$W_{-1/3}(\text{spo}(2|3), f_0) \cong W^{-1/3}(\text{spo}(2|3), f_0)/I$$

contains the difference $T = L - L_{\text{sug}}$, where

$$L_{\text{sug}} = \frac{3}{16} j^0 j^0 + \frac{3}{4} j^+ j^- - \frac{3}{8} j^0,$$

is the conformal field of the $\mathfrak{sl}_2$ sub-VOA. Further direct calculations such as
\begin{align}
W^{+,+}(0,1) &= -2T_1W^{+,+}(0,1) - \frac{3}{2}z^jW^{+,+}(0,0) - \frac{3}{32}\frac{1}{w}\partial j^+ j^+ \\
&= \frac{1}{4}\frac{1}{w}\partial j^+ j^+ + \frac{1}{8}\frac{1}{w}\partial^2 j^+ j^+
\end{align}

(5.8)

and

\begin{align}
W^{0,+}(0,0) &= \frac{2}{3}T_1W^{0,+}(0,0) - \frac{1}{12}z^0 j^+ j^+ - \frac{1}{12}z^0 j^+ L^+ \\
&= -\frac{7}{48}\frac{1}{w}\partial j^+ j^+ + \frac{7}{72}\partial^2 j^+
\end{align}

(5.9)

allow us to remove the fields \(W^{i,j}(a, b)\) from the generating set described in Theorem 5.1 in this setting leading to the following.

As a consequence we obtain a different proof of result obtained earlier in [AKMPP, Theorem 6.8]:

**Corollary 5.1.** We have

\[ \mathcal{W}_{-1/3}(\mathfrak{sp}(2|3), f_0)^{\mathbb{Z}_2} \cong L_{-2/3}(\mathfrak{sl}_2). \]

5.2. The orbifold \( \mathcal{W}^k(\mathfrak{sp}(2|3), f_0) \otimes \mathcal{F}^{\mathbb{Z}_2} \). The \( N = 3 \) algebra \( \mathcal{W}^k(\mathfrak{sp}(2|3), f_0) \) is often considered tensored with a rank one fermion algebra \( \mathcal{F} \), generated by one odd field \( \varphi \) with OPE given by

\[ \varphi(z)\varphi(w) \sim -\frac{1}{2}(2k + 1)\frac{1}{z - w}. \]

(5.10)

The automorphism \( \Psi \) described in (5.2) can be extended to this setting by imposing

\[ \Psi(\varphi) = -\varphi. \]

(5.11)

It is clear in the this case the orbifold inherits the generators described by (5.4) in addition to the fields

\[ \omega_j(a, b) = \varphi^a \varphi^b G_j^a \quad \text{and} \quad \omega_\varphi(a, b) = \varphi^a \varphi^b \varphi_j^a, \]

where \( j \in \{0, +, -\} \) and \( a, b \geq 0 \). In this setting the expressions (5.5) will have their usefulness superseded by the expressions of lower weight

\[ \omega_i(a_0, b_0)\omega_j(a_1, b_1) + \omega_i(a_0, b_0)\omega_j(a_1, b_1), \]

and

\[ \omega_i(a_0, b_0)\omega_\varphi(a_1, b_1) + \omega_i(a_0, b_0)\omega_\varphi(a_1, b_1), \]

for \( a_0, a_1, b_0, b_1 \geq 0 \) and \( i, j \in \{0, +, -\} \).

**Theorem 5.2.** For \( k \neq -\frac{1}{2} \) the orbifold \( \mathcal{W}^k(\mathfrak{sp}(2|3), f_0) \otimes \mathcal{F}^{\mathbb{Z}_2} \) is minimally generated by the fields \( j^0, j^+ L, W^{0,0}(0,0), W^{+, -, -}(0,0), \omega\varphi(0,1), \omega_\varphi(0,1), \omega_\varphi(0,1), \omega_\varphi(0,1), \) and \( \omega(0,1) \) and is of type \( (1^3, 2^4, 3^6) \).

**Proof.** It follows from the well-known fact that \( \mathcal{F}^{\mathbb{Z}_2} \cong L(-\frac{1}{2}, 0) \) that the generators of the form \( \omega_\varphi(a, b) \) can be reduced to the single field \( \omega_\varphi(0,1) \).

Next, we focus on the fields of the form \( \varphi^{i,j}(a, b) \) where \( i, j \in \{0, +, -\} \), with explicit calculations given for the case when \( i = j = 0 \) as the others follow similarly. An argument similar to that found in [CL] and [CPS], using the translation operator, allows us to reduce to the fields \( W^{i,j}(0, m) \) for \( m \geq 0 \) with \( m \notin 2\mathbb{Z} \) if \( i = j \). Finally, quantum corrections of (5.13), such as
\[(1 + 2k)W^{0,0}(0, m + 1) = -2^3\omega_0(0, 0)\omega_0(0, m)^* - (1 + 2k)\partial W^{0,0}(0, m)\]
\[\quad + \frac{(1 + 2k)(3 + 4k)}{8m + 12} \omega_0(0, m + 3)\]
\[\quad + 2 \sum_{\ell=0}^m \frac{1}{\ell + 1} \left( \frac{m}{m - \ell} \right) \omega_0(0, m + 3)\]
\[\quad \omega_0(0, m + 3) \omega_0(0, m + 3)\]
for \(m \geq 0\) with \(A = -\frac{1}{2}(1 + 2k)L - \frac{1}{16} j^0 j^0\), allow us to eliminate all fields of the form \(W^{0,0}(0, m + 1)\) from our generating set, leaving us with only the fields \(W^{0,0}(0, 0), W^{0,0}(0, 0), \) and \(W^{0,0}(0, 0)\) remaining from this family of generators.

Now we move on to the generators of the form \(\omega_i(a, b)\) where \(i \in \{0, +, -\}\) and \(a, b \geq 0\). Again, as above, we may only consider the fields of the form \(\omega_i(0, m)\) where \(m \geq 0\). A quantum correction of (5.14), namely,
\[\omega_i(0, m + 1) = \frac{4}{3} \omega_i(0, 0)\omega_i(0, 1) + \frac{2}{3} (1 + 2k)\partial \omega_i(0, m + 1),\]
which holds for all \(m \geq 0\), eliminates the generators \(\omega_i(0, m + 2)\), leaving only \(\omega_i(0, 0)\) and \(\omega_i(0, 1)\).

\[\square\]

Now we move towards our main result, which will follow from Corollary 5.1 and Theorem 5.2. Again, we set \(k = -\frac{1}{2}\) and now consider the algebra \(W_{-1/3}(\mathfrak{spo}(2|3), f_0) \otimes \mathcal{F}\), which inherits the singular vector, \(T\), described above (5.7), from the corresponding algebra without the free fermion. In this setting, we begin with the generating set described by Corollary 5.1 with the addition of the fields \(\omega_j(0, 0), \omega_j(0, 1), \) and \(\omega_0(0, 1)\) for \(j \in \{0, +, -\}\). Now, equations such as
\[\omega_+(0, 1) = T_0\omega_+(0, 0) + \frac{3}{4} j^0 \omega_+(0, 0) - \frac{3}{4} j^+ \omega_0(0, 0)^*,\]
will allow us to remove the fields \(\omega_j(0, 1)\) for \(j \in \{0, +, -\}\). Observe that we may replace the generator \(\omega_0(0, 1)\) with the field
\[\bar{L} = L_{\text{aug}} + 3\omega_0(0, 1),\]
setting up the following result, which is the main result of this section.

**Theorem 5.3.** The orbifold \((W_{-1/3}(\mathfrak{spo}(2|3), f_0) \otimes \mathcal{F})^{\mathbb{Z}_2}\) has a minimal strong generating set of the fields \(j^0, j^\pm, \bar{L}, \omega_0(0, 0), \) and \(\omega_\pm(0, 0)\) and is of type \((1, 1, 1, 2, 2, 2)\). Moreover, we have
\[(W_{-1/3}(\mathfrak{spo}(2|3), f_0) \otimes \mathcal{F})^{\mathbb{Z}_2} \cong W_{-7/3}(\mathfrak{sl}_4, f).\]

**Proof.** It is straightforward to check that the identification
\[(W_{-1/3}(\mathfrak{spo}(2|3), f_0) \otimes \mathcal{F})^{\mathbb{Z}_2} \rightarrow W_{-7/3}(\mathfrak{sl}_4, f),\]
defined by
\[j^0 \mapsto J^0,\quad j^\pm \mapsto J^\pm,\]
\[\bar{L} \mapsto L,\quad \omega_0(0, 0) \mapsto -\frac{i}{2\sqrt{6}} H,\]
\[\omega_+(0, 0) \mapsto \frac{i}{2\sqrt{6}} E,\quad \omega_-(0, 0) \mapsto -\frac{i}{2\sqrt{6}} F,\]
preserves OPE and since \((W_{-1/3}(\mathfrak{spo}(2|3), f_0) \otimes F)_{\mathbb{Z}_2}\) is simple this defines a surjective (and thus bijective) vertex algebra homomorphism. □

References

[A] D. Adamović, Realizations of simple affine vertex algebras and their modules: the cases \(\hat{\mathfrak{sl}}_2\) and \(\hat{\mathfrak{osp}}(1,2)\), Communications in Mathematical Physics, March 2019, Volume 366, Issue 3, pp 1025-1067; arXiv:1711.11342 [math.QA]


[CPS] O. Chandrasekhar, M. Penn, and H. Shao, \(Z_2\) invariants of the rank \(n\) free fermion algebra, Comm. Alg. to appear


[FZ] Frenkel - Ben-Zvi


