(1) Find all value(s) of $h$ for which the system

\begin{align*}
  x_1 + 3x_2 - x_3 &= 5 \\
  -x_1 + x_2 + x_3 &= 1 \\
  x_1 + 2x_2 - 3x_3 &= 0 \\
  -x_1 - x_2 - x_3 &= h
\end{align*}

is consistent.

(2) Solve the linear system

\begin{align*}
  2x_1 + 3x_2 - x_3 - 4x_5 &= 1 \\
  -x_1 + 2x_3 - x_4 + 2x_5 &= 0 \\
  x_1 + x_2 - x_3 - x_4 - x_5 &= 1
\end{align*}

by bringing the augmented matrix of the system to the reduced row-echelon form. Write the solution as $p + X_h$ where $X_h$ is the solution of the corresponding homogeneous system.

(3) Which of the following is true/false?

(a) Every homogeneous linear system is consistent.
(b) If a set of vectors $\{u, v, w\}$ is linearly independent, then $\{u, v\}$ is linearly independent as well.
(c) Three vectors in $\mathbb{R}^3$ are linearly dependent if and only if they are contained on the same line.
(d) Any linear system has either none, one, two, or infinitely many solutions.
(e) If $A$ is a $2 \times 3$ matrix and $B$ a $3 \times 2$ matrix, then $(B \cdot A) \cdot (A \cdot B)$ is well-defined.
(f) The matrix equation $Ax = b$ has a solution if and only if $b$ is in the span of the column vectors of $A$.
(g) Every set of 10 vectors in $\mathbb{R}^{11}$ is linearly independent.
(h) Product of two invertible matrices of the same size is an invertible matrix.
(i) Let $T_A$ and $T_B$ be linear transformations on $\mathbb{R}^n$ with standard matrices $A$ and $B$, respectively. Then the standard matrix of $T = T_A \circ T_B$ is $B \cdot A$. (j) If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $AB = 0$ (zero matrix). (k) A set of 5 vectors in $\mathbb{R}^5$ that spans $\mathbb{R}^5$ must be also linearly independent.

(4) Find a matrix form of the linear transformation that reflects every vector in $\mathbb{R}^3$ through the $xz$-plane.

(5) The linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by the matrix

$$
\begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}.
$$
Draw a triangle with vertices (1,1), (2,0) and (0,0) and determine where it is being mapped under the transformation $T$.

(6) Describe the image of the triangle in Problem 5 under the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ which first rotates points through $\frac{\pi}{4}$ and then reflects through the $x$-axis. Write the standard matrix of $T$.

(7) Determine all vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ that are inside

\[
\text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix} \right\}.
\]

(8) Is there a linear transformation $\mathbb{R}^3$ to $\mathbb{R}^4$ which is

(i) one-to-one?
(ii) onto?

Explain your reasoning in each case.

(9) Determine whether the vectors

\[
\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}
\]

are linearly independent.

(10) Find the inverse of

\[
\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

(11) Let

$T(x, y, z) = (-x + y + z, 2x - 2z, y + 2z)$.

Find $A$, such that $T(x) = Ax$. Show that $T$ is an invertible transformation. Finally, fill in the missing slots

$T^{-1}(x, y, z) = (\quad , \quad , \quad )$.

(Hint: The standard matrix of the inverse of $T$ is precisely $A^{-1}$)

(12) Give at least 5 different characterizations of invertible matrices.
(13) Find the standard matrix of \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) if \( T \) rotates points by the angle \( \frac{2\pi}{3} \) counterclockwise.

(14) (a) Compute

\[
\begin{vmatrix}
    a & b & c & d \\
    3e & 3f & 3g & 3h \\
    i - 2a & j - 2b & k - 2c & l - 2d \\
    m & n & o & p
\end{vmatrix}
\]

if

\[
\begin{vmatrix}
    p & o & n & m \\
    e & k & j & i \\
    h & g & f & e \\
    d & c & b & a
\end{vmatrix} = 2.
\]

(b) Use row reduction method to compute the determinant

\[
\begin{vmatrix}
    2 & -2 & -6 & 4 \\
    2 & 2 & 4 & 4 \\
    -3 & 3 & 9 & 6 \\
    2 & 2 & -2 & 2
\end{vmatrix}
\]

(15) Consider a shear linear transformation \( T \) in \( \mathbb{R}^2 \) given by the matrix

\[
\begin{bmatrix}
    1 & 2 \\
    0 & 1
\end{bmatrix}
\]

Let \( \Delta \) be a triangle with vertices at \((-1, 0)\), \((1, 0)\) and \((0, 1)\). What is the area of the triangle obtained from \( \Delta \) if we apply the shear transformation six times in a row?

(Extra Credit) Let \( A \) be an \( n \times n \) matrix such that \( A^2 \). Show that \( I - A \) is invertible.

(Extra Credit) Describe all \( 2 \times 2 \) matrices \( X \) for which

\[ AX =XA, \]

for every \( 2 \times 2 \) matrix \( A \).