MOCK FINAL EXAM, Math 220

Name:

(i) (8 pts)
(a) Determine $\lambda$ and $\nu$ such that the system

\[-x_1 + 2x_2 - x_3 = 2 \\
3x_1 + x_2 + \lambda x_3 = 2 \\
2x_1 - 3x_2 + x_3 = \nu
\]

has: (a) no solution, (b) a unique solution, and (c) infinitely many solutions. Solve the
system when there are inf. many solutions.

Solution: (a) $\lambda = -4, \nu \neq -20/7$
(b) $\lambda \neq -4$.
(c) $\lambda = -4$ and $\nu = -20/7$. In this case the solution is given by

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  2/7 \\
  8/7 \\
  0
\end{bmatrix} + x_3 \begin{bmatrix}
  1 \\
  1 \\
  1
\end{bmatrix}
\]

(ii) Define the null space, the column space, and the rank of a matrix $A$. Explain why
invertible $n \times n$ matrices are always of rank $n$ and $\det(A) \neq 0$

Solution: The null space of $A$ is the vector space of solutions of the matrix equation
$Ax = 0$. The column space $Col(A)$ is defined as the span of the columns of $A$. The rank
of $A$ is the dimension of the column space. Invertible matrices are of rank $n$ because
(for example) they are row equivalent to the identity matrix so they have pivot in each
column. $\det(A) \neq 0$ follows (for instance) from the formula $\det(A)\det(A^{-1}) = 1$ so it
must be nonzero.
(iii) (a) Explain why an orthogonal matrix must have orthonormal columns.
(b) Use (a) to fill in the missing entries so that
\[ A = \begin{bmatrix} \frac{1}{\sqrt{2}} & ? & -\frac{1}{\sqrt{2}} \\ ? & ? & \frac{1}{\sqrt{2}} \\ 0 & ? & ? \end{bmatrix} \]
is orthogonal.
(c) How many orthogonal matrices as in (b) are there?
(d) Find the inverse of the matrix that you computed in part (b).

Solution: Solved in class.
(iv) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$,

$$T(x_1, x_2, x_3) = (3x_1 - x_2, 6x_1 - 2x_2, x_3).$$

Find a basis of the kernel of $T$ and a basis of the range of $T$.

Solution:

$Ker(T)$ is one dimensional with basis $(1, 3, 0)^T$, while $Range(T)$ is 2-dimensional with basis: $\{(3, 6, 0)^T, (0, 0, 1)^T\}$.
(v) Argue that $H$ is a vector space. Find a basis of $H$ in each case.

(a) Let $H = \left\{ \begin{bmatrix} a-b \\ b-c \\ a-d \\ d-c \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \subset \mathbb{R}^4$

**Solution:** This question is equivalent to finding a basis of $\text{Col}(A)$ where

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Easy row reductions gives a basis: the first three columns in $A$.

(b) $H = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : 2a + 4c + 2d = 0, a + b - 3c + d = 0 \right\} \subset \mathbb{R}^4$

**Solution:** Basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$
(vi) Let $S$ be the triangle determined by the vectors \[
\begin{bmatrix}
-2 \\
3
\end{bmatrix}
\] and \[
\begin{bmatrix}
1 \\
4
\end{bmatrix}
\]. Let $T(x_1, x_2) = (6x_1 - 2x_2, -3x_1 + 2x_2)$. Find the area of the image of $S$ under $T^{-1}$ (the inverse of $T$).

Solution: Solved in class.
(vii) Diagonalize the matrix

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}. \]

Compute the characteristic polynomial of \( A \) and determine its eigenvalues. If \( A \) is diagonalizable find \( P \) and \( D \) such that \( A = PDP^{-1} \).

**Solution:** The matrix is diagonalizable.

\[
P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.
\]
(viii) Let $B = \{ \begin{bmatrix} 1 \\ -2 \\ \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix} \} \text{ and } C = \{ \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ \end{bmatrix} \}$. 

(a) Compute $[v]_B$ if $v = \begin{bmatrix} 3 \\ 3 \\ \end{bmatrix}$; (this is $v$ in the standard basis, not $[v]_C$!).

(b) Compute $P_{B\rightarrow C}$

Solution:

(a) $[v]_B = P_B^{-1} \cdot v = \begin{bmatrix} 0 \\ 3 \\ \end{bmatrix}$

(b) $P_{B\rightarrow C} = P_B^{-1} P_C = \begin{bmatrix} -1/3 & -1/3 \\ 1/3 & 7/3 \\ \end{bmatrix}$
(ix) (a) Use the method of row-reduction to compute the determinant

\[
\begin{vmatrix}
0 & 2 & 3 & 0 \\
3 & 3 & 3 & 3 \\
2 & 4 & 6 & 6 \\
4 & 2 & 4 & 2 \\
\end{vmatrix}
\]

**Solution:** $Det = 0$.

(b) Compute

\[
\begin{vmatrix}
a + b & a - 2b & a + 3b \\
a + c & a - 2c & a + 3c \\
a + d & a - 2d & a + 3d \\
\end{vmatrix}
\]

Show your work!

**Solved in class:** $det = 0$

(c) Compute $det(3B^T A^T C^{-1})$ if $det(AB) = 6$, and $det(C) = 2$, where $A$, $B$ and $C$ are $2 \times 2$ matrices. Explain your reasoning!

**Solution:**

\[
(c) det(3B^T A^T C^{-1}) = 3^2 \cdot det(AB) \cdot \frac{1}{det(C)} = 9 \cdot 6 / 2 = 27.
\]
(x) Use the Gram-Schmidt process to find an orthogonal basis for the subspace \( W \) of \( \mathbb{R}^4 \) spanned by \({ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \}). Let \( y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \). Compute the orthogonal projection of \( y \) on \( W \) and the shortest distance from \( W \) to \( y \).

**Solution:** We have to find an orthogonal basis of \( W \) first. By Gram-Schmidt we get

\[
\begin{align*}
\mathbf{u}_1 &= \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \\
\mathbf{u}_2 &= \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}
\end{align*}
\]

By using the formula

\[
\hat{y} = \frac{\mathbf{u}_1 \cdot y}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot y}{\mathbf{u}_2 \cdot \mathbf{u}_1} \mathbf{u}_2
\]

After some computation we get \( \hat{y} = y \).

Conclusion: the vector \( y \) is already in \( W \) so the distance is \( ||y - \hat{y}|| = 0 \).

n.b. I realized that in the previous version I had \( y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \). With this choice of \( y \), we get

\[
\hat{y} = \frac{\mathbf{u}_1 \cdot y}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{u}_2 \cdot y}{\mathbf{u}_2 \cdot \mathbf{u}_1} \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},
\]

\[
||y - \hat{y}|| = \sqrt{2}
\]

(xi) Find the least-squares solution of \( Ax = b \), where \( A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{bmatrix} \), \( b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \).

**Solution:** You have to solve \( A^T A \mathbf{x} = A^T b \). The answer is \((x_1, x_2) = (5/7, 1/7)\).
Extra Credit:

(i) Let $A$ and $B$ be any $n \times n$ matrices. Show that $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

(ii) The link matrix of a web is given by

\[
\begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 0
\end{bmatrix}
\]

Draw the link diagram for this web. By using Google PageRank algorithm, compute its importance score vector.