Notes on dynamic stability of steady-states of Difference equation systems

In general we can have

$$F(x_t, x_{t-1}) = \mathbf{0}$$

where x is a k-dimensional vector, $\mathbf{0}$ is the k-dimensional nul-vector and $F : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k$. A Steady state is an x^* such that

$$F(x^*, x^*) = 0$$

First we need to linearize the system around x^*. To do this implicitly differentiate F at x^*:

$$\sum_{i=1}^{k} \frac{\partial F^j}{\partial x_t^i} dx_t^i + \frac{\partial F^j}{\partial x_{t-1}^i} dx_{t-1}^i = 0 \quad \text{for } j = 1..k. \tag{1}$$

where F^j represents the jth component of F and x_t^i is the ith component of x_t. We want to write the linearized system in the form

$$x_t - x^* = H^* \left(x_{t-1} - x^* \right) \tag{2}$$

where H is a square matrix of the form

$$\begin{bmatrix} \frac{dx_t^1}{dx_{t-1}^1} & \frac{dx_t^1}{dx_{t-1}^2} & \cdots & \frac{dx_t^1}{dx_{t-1}^k} \\ \frac{dx_t^2}{dx_{t-1}^1} & \ddots & & \\ \vdots & & \ddots & \\ \frac{dx_t^k}{dx_{t-1}^1} & & & \frac{dx_t^k}{dx_{t-1}^k} \end{bmatrix}$$

evaluated at $x = x^*$. The components of H are obtained from (1) by setting each of the dx_{t-1}^i except one equal to zero, dividing through by that non-zero differential, evaluating at $x_t = x_{t-1} = x^*$ and solving.

If the original system can be written in the simpler form $x_t = G(x_{t-1})$ then (2) represents the first order Taylor series approximation to $G(.)$ around x^*. H^* is then simply the Jacobean of G evaluated at $x_{t-1} = x^*$.

If all we are interested in is the stability properties of the system at x^*, once H^* has been obtained (2) can be rewritten by setting $y_t \equiv x_t - x^*$ so that $y_t = H^* y_{t-1}$.

Now as long as H^* has nondegenerate eigenvalues and corresponding linearly independent eigenvectors it can be decomposed in the usual way (see http://mathworld.wolfram.com/EigenDecomposition.html) into

$$H^* = PDP^{-1}$$

where P is the $k \times k$ matrix made up the eigen vectors and D is the diagonal matrix with eigen values of H on the diagonal. By iterating the equation $y_t = Hy_{t-1}$ backwards we get

$$y_t = H^{*t} y_0$$

where y_0 is any starting value for the linearized system. Now

$$H^{*t} = \left(PDP^{-1}\right)^t = PD^tP^{-1}$$

and for any diagonal matrix raising it to a power simply generates another diagonal matrix with the diagonal elements of the original matrix raised to that power. So

$$D^t = \begin{bmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \lambda_k^t \end{bmatrix}$$

where the λ_i's are the eigen values of H^*. Then if $|\lambda_1| < 1$ that element of D^t disappears for large values of t. $|\lambda_1| > 1$ the element becomes very large. Thus the system is unstable in the directions indicated by the eigen vectors associated with the eigen values that are larger then 1 in absolute value. When ever $|\lambda_1| \neq \lambda_1$ then the the system is oscillatory on the implied direction.

Note: Not all matrices, H^* will have nondegenerate eigenvalues and corresponding linearly independent eigenvectors. In those cases typically correspond to the non-generic parameter configurations that generate bifurcations.