06E:204 Macroeconomics
Suggested solutions to Assignment 2

Problem 1.

In this representative agent model we do not have any distortions (taxes are lump-sum) and thus, the First Welfare Theorem holds. The competitive equilibrium (C.E.) is Pareto optimal (P.O.), therefore we can solve social planner's problem for competitive allocation..

Social planner's problem: Social planner maximizes agent's utility subject to the resource constraint in the economy.

$$\max_{c,l} \; u(c,l) \tag{1}$$
$$s.t.\; c+g = zf(k_0,n) \tag{2}$$
$$l+n = 1 \tag{3}$$

Given the utility function's properties we can substitute (2) and (3) into the utility function (1). Note that the agent will supply all his endowment of capital, since the utility is increasing in consumption. Then we apply the first order conditions with respect to l. Note that the social planner takes government expenditures as given.

The FOC:

$$l: \quad -zf_2(k_0,1-l)u_1(zf(k_0,1-l)-g,l)+u_2(zf(k_0,1-l)-g,l)=0 \tag{4}$$

Given the actual functional form of the utility and the production, we can use (4) to solve for leisure l, and consequently for n, c, y using (2) and (3). We use firm's profit maximization problem to pin down the *competitive* real wage and capital rental rate:

the FOC:

$$n \;:\quad w=\frac{\partial y}{\partial n}=zf_2(k_0,n) \tag{5}$$
$$k \;:\quad r=\frac{\partial y}{\partial k}=zf_1(k_0,n)$$

To determine the equilibrium effects of a change in government purchases on leisure we have to totally differentiate (4) and solve for $\frac{dl}{dg}$:

$$\frac{dl}{dg}=\frac{u_{21}-zf_2u_{11}}{z^2f_2^2u_{11}-2zf_2u_{12}+u_{22}+zu_1f_{22}}<0 \tag{6}$$

Note that the denominator is negative because $z^2f_2^2u_{11}-2zf_2u_{12}+u_{22}<0$ due to the quasi-concavity of $u(\cdot)$ (the bordered Hessian of the utility function is negative semidefinite), $u_1>0$ because the utility is increasing in consumption and we also assume that the production technology exhibits diminishing marginal returns to labor and capital and then $f_{22}<0$.

Since consumption and leisure are normal goods in our economy, then $u_{21}-zf_2u_{11}>0$. Thus the result in (6). Also

$$\frac{dn}{dg} = \frac{d(1-l)}{dg} = -\frac{dl}{dg} = -\frac{u_{21} - zf_2u_{11}}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} > 0 \tag{7}$$

The employment respectively, increases as the government increases its expenditures.

The wage will respond in a following way to an increase in government spending:

$$\frac{dw}{dg} = \frac{d}{dg}(zf_2(k_0, 1-l)) = -zf_{22}\frac{dl}{dg} < 0$$

So, if the government increases its spending then this produces an increase in taxes and wages. Thus, the consumer's income goes down. Therefore, there is a pure income effect. Because consumption and leisure are normal goods, their quantities will go down.

$$\begin{aligned} \frac{dc}{dg} &= \frac{d(zf(k_0, 1-l) - g)}{dg} = \\ &= -zf_2\frac{dl}{dg} - 1 = \frac{zf_2u_{12} - u_{22} - zu_1f_{22}}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} < 0 \end{aligned} \tag{8}$$

The result in (8) follows from the fact that consumption and leisure are normal goods and therefore, $zf_2u_{12} - u_{22} > 0$, and also $zu_1f_{22} < 0$.

$$\begin{aligned} \frac{dy}{dg} &= -zf_2\frac{dl}{dg} = \\ &= \frac{-zf_2(u_{21} - zf_2u_{11})}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} > 0 \end{aligned} \tag{9}$$

Due to the result that employment n increases as g increases, the out put will also increase. Here we have crowding out of the agent's consumption by the government spending (and thus, a decrease in consumption is smaller than an increase in government spending). Also note that from (8) and (9) we have

$$\frac{dy}{dg} < 1$$

i.e., the 'balanced budget multiplier" is less than 1.

(b) To determine the equilibrium effects of a change in total factor productivity on consumption, employment, the real wage and output we totally differentiate (4) and solve for $\frac{\mathrm{dl}}{\mathrm{dz}}$:

$$\frac{dl}{dz} = \frac{-f(u_{21} - zf_2u_{11}) + f_2u_1}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} \tag{10}$$

In (10) we cannot sign the nominator of the expression. Therefore, we cannot say whether leisure increases or decreases with an increase in total factor productivity. This is due to the opposite income and substitution effects. Before going into details about these effects, we will look how the real wage changes when z increases (see (4)):

$$\frac{dw}{dz} = f_2 - zf_{22}\frac{dl}{dz} = \frac{(zf_{22}f - zf_2^2)(u_{12} - zf_2u_{11}) - f_2(zf_2u_{12} - u_{22})}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} > 0$$

Thus, an increase in total factor productivity will produce an increase in the real wage. Note that $u_{12} - zf_2u_{11} > 0$ and $zf_2u_{12} - u_{22} > 0$, assuming consumption and leisure are normal goods. Also $f_2 > 0$ (production function is increasing in inputs).

Since an increase in z increases the real wage, the agent experiences an increase in income. But consumption and leisure are normal goods, therefore, their quantities must go up (income effect). However, as the real wage is a price for leisure, its increase leads to the increased price of leisure, and the agent would like to substitute relatively more expensive good ”leisure” for ”consumption”, relatively cheaper good (substitution effect).

To decompose $\frac{dl}{dz}$ into $\frac{dl}{dz}(subst.)$ and $\frac{dl}{dz}(income)$ we first find $\frac{dl}{dz}(subst.)$. To find the substitution effect on leisure we vary prices and see how much we should compensate the agent to keep his utility constant. We totally differentiate (11) and (12 equivalent to 4) and solve for $\frac{dl}{dz}(subst.)$.

$$u(c,l) = \overline{u} \tag{11}$$

where $\overline{u}$ is a constant,

$$-zf_2(k_0, 1-l)u_1(c,l) + u_2(c,l) = 0 \tag{12}$$

and use the condition that in equilibrium we have:

$$w = \frac{u_2}{u_1} = zf_2(k_0, n)$$

Then we have

$$\frac{dl}{dz}(subst.) = \frac{f_2u_1}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} < 0 \tag{13}$$

and then

$$\frac{dl}{dz}(income) = \frac{dl}{dz} - \frac{dl}{dz}(subst.) = \frac{-f(u_{21} - zf_2u_{11})}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} > 0 \tag{14}$$

However, we do not know the exact value of both effects, thus we cannot determine which one will dominate for leisure.

There will be the following effect on employment:

$$\begin{aligned}
\frac{dn}{dz} &= -\frac{dl}{dz} = \\
&= -\frac{-f(u_{21} - zf_2u_{11}) + f_2u_1}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} = \\
&= \frac{f(u_{21} - zf_2u_{11}) - f_2u_1}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} \qquad (15)
\end{aligned}$$

and its sign is indeterminate. We could also decompose (15) into income and substitution effects, but the sign of $\frac{dn}{dz}$ will be undetermined as well.

With regard to consumption and output:

$$
\begin{aligned}
\frac{dc}{dz} &= \frac{dy}{dz} = \frac{d(zf(k_0, 1-l)-g)}{dz} = f - zf_2\frac{dl}{dz} = \\
&= \frac{-f(zf_2u_{12}-u_{22}) + zu_1f_{22}f - zf_2^2u_1}{z^2f_2^2u_{11} - 2zf_2u_{12} + u_{22} + zu_1f_{22}} > 0 \qquad (16)
\end{aligned}
$$

From (16) it is clear that output and consumption will increase as z increases. Though labor input can either decrease or increase, an increase in total factor productivity dominates even negative movement in labor input.

To summarize:

When there is an increase in total factor productivity, it has dual effect on leisure and thus employment: (a) income effect: the real wage rises as z increases, as the result, available to the agent income increases; plus, leisure is a normal good; therefore, the agent increases his leisure (decreases his employment); (b) substitution effect: as the real wage increases, it's more expensive to enjoy leisure, thus the agent want to substitute leisure for relatively less expensive consumption, and leisure decreases.

The increased total factor productivity dominates income effect in leisure choice: the output and consumption will increase no matter whether income or substitution effect dominates in the leisure choice.