AECO601-Macroeconomics I Fall 2008
Suggested solutions to Assignment 1

Problem 1.

This problem can be solved in two ways. We can either directly solve for competitive equilibrium allocations or solve Social planner's problem because the First Welfare Theorem holds in this case. Here we use the first approach.

(a) To find a competitive equilibrium (C.E.) allocation we need to solve for c, l, n, k and prices w and r, such that

1. the consumer chooses c and l to maximize his utility given w and r;
2. the representative firm chooses n and k to maximize its profit given w and r;
3. markets for labor, capital and consumption good clear.

{1} Firm's problem: the firm chooses labor n and capital k to maximize its profit treating wage w and rental rate r as given. That is, the firm solves

$$\max_{n,k} zk^{\alpha}n^{1-\alpha} - wn - rk$$

The first order conditions (FOC's) are:

$$n \quad : \quad w = \frac{\partial y}{\partial n} = (1-\alpha)zk^{\alpha}n^{-\alpha} \tag{1}$$

$$k \quad : \quad r = \frac{\partial y}{\partial k} = \alpha zk^{\alpha-1}n^{1-\alpha} \tag{2}$$

{2} Consumer's problem:

The consumer maximizes his utility subject to his budget constraint taking wage w and rental rate r as given. That is, the consumer solves

$$\begin{aligned}
\max_{c,l,k} u(c,l) &= \max_{c,l,k} c + \beta l && (3)\\
s.t. \quad c &\leq w(1-l) + rk && (4)\\
0 &\leq l \leq 1 && (5)\\
k &\leq k_0 && (6)\\
c &\geq 0 && (7)
\end{aligned}$$

Constraints (4) and (6) hold with equality because $u_c(\cdot,\cdot)$, $u_l(\cdot,\cdot) > 0$ (More is preferred to less). Thus,

$$\max_{l} w(1-l) + rk_0 + \beta l = \max_{l} w + rk_0 + (w-\beta)l$$

There are three cases:

1) if $\beta > w$, then $l = 1, n = 0$ and $w \to \infty$ (please, see (1)), and no output is produced.

2) if $\beta < w$, then$l = 0, n = 1$. This case can be supported if $(1-\alpha)\beta^{-1} z k_0^{\alpha} > 1$ (i.e. $\beta < w$).

Given the equilibrium condition that markets clear (condition 3), we can solve for

$$\begin{aligned} l &= 0, n = 1 & (8) \\ y &= z k_0^{\alpha} & (9) \\ c &= w + r k_0 = z k_0^{\alpha} & (10) \\ w &= (1-\alpha) z k_0^{\alpha} & (11) \\ r &= \alpha z k_0^{\alpha - 1} & (12) \end{aligned}$$

3) if $\beta = w$,then $l \in [0, 1]$. Then, given that markets clear at the equilibrium (condition $\{3\}$), we have

$$\begin{aligned} w &= (1-\alpha) z k_0^{\alpha} n^{-\alpha} = \beta \Rightarrow n = [(1-\alpha)\beta^{-1} z]^{\frac{1}{\alpha}} k_0 & (13) \\ l &= 1 - n = 1 - [(1-\alpha)\beta^{-1} z]^{\frac{1}{\alpha}} k_0 & (14) \\ y &= c = z^{\frac{1}{\alpha}} \{[(1-\alpha)\beta^{-1}]^{\frac{1-\alpha}{\alpha}} k_0\} & (15) \\ w &= \beta & (16) \\ r &= \alpha z^{\frac{1}{\alpha}} [(1-\alpha)\beta^{-1}]^{\frac{1-\alpha}{\alpha}} & (17) \end{aligned}$$

This can be supported if $(1-\alpha)\beta^{-1} z k_0^{\alpha} \leq 1$ (i.e., $n \leq 1$).

$\{3\}$ Two market clearing conditions are satisfied at the equilibrium (we can drop the third one because of the Walras' Law):

consumption good market clears:

$$y = c$$

labor market clears:

$$n + l = 1$$

Please, see in (8)-(17) how we used these conditions in solving for competitive allocations.

(b) Here we look at the differential change in the productivity z. Differentiate (8)-(18) with respect to z.

case 2)

$$\frac{\partial l}{\partial z} = \frac{\partial n}{\partial z} = 0 \tag{18}$$

$$\frac{\partial y}{\partial z} = \frac{\partial c}{\partial z} = k_0^\alpha > 0 \tag{19}$$

$$\frac{\partial w}{\partial z} = (1-\alpha)k_0^\alpha > 0 \tag{20}$$

$$\frac{\partial r}{\partial z} = \alpha k_0^{\alpha-1} > 0 \tag{21}$$

If the productivity parameter z increases, then this leads to an increase in output, wage and rental rate. This produces an increase in income available to the consumer, and thus, consumption will also increase (assuming it's a normal good). However, leisure and employment will be unchanged.

case 3)

$$\frac{\partial l}{\partial z} = -\frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} [(1-\alpha)\beta^{-1}]^{\frac{1}{\alpha}} k_0 < 0 \tag{22}$$

$$\frac{\partial n}{\partial z} = \frac{1}{\alpha} z^{\frac{1-\alpha}{\alpha}} [(1-\alpha)\beta^{-1}]^{\frac{1}{\alpha}} k_0 > 0 \tag{23}$$

$$\frac{\partial y}{\partial z} = \frac{\partial c}{\partial z} = \frac{1}{\alpha}\{[(1-\alpha)\beta^{-1} z]^{\frac{1-\alpha}{\alpha}} k_0\} > 0 \tag{24}$$

$$\frac{\partial w}{\partial z} = 0 \tag{25}$$

$$\frac{\partial r}{\partial z} = [(1-\alpha)\beta^{-1} z]^{\frac{1-\alpha}{\alpha}} > 0 \tag{26}$$

In this case, an increase in productivity produces the following effects: leisure decreases as agents want to take an advantage of the increased productivity to produce more of consumption good, thus employment increases. Although the wage is constant, income increases through the increased employment hours and thus, the consumption will increase. Rental rate of capital also increases because of an increase in marginal productivity of capital.

Problem 2.

(a) Pareto Optimum (P.O.): To find P.O. allocation we solve the social planner's problem. The social planner maximizes consumer's utility subject to the resource constraint in the economy. Note, that the social planner is not concerned about the prices w and r. Also note that, at the optimum, the average consumption of the agents is c because the total mass of consumers is 1 and symmetry condition on allocations is applied. Notice, the social planner does not take the average consumption level as given.

Social planner solves:

$$
\begin{aligned}
\max_{c,l} U(c,l,\overline{c}) &= \max_{c,l} \; u(c,l)+v(\overline{c}) \\
s.t. \; y &= c \\
y &= n \\
n+l &= 1 \\
\overline{c} &= c
\end{aligned}
$$

After the substitution of the constraints into the utility function, the problem becomes

$$\max_{l} u(1-l,l)+v(1-l)$$

Then, the FOC is:

$$-u_1(1-l,l)+u_2(1-l,l)-v\acute{}(1-l)=0 \tag{1}$$

Given an actual functional form for the utility function, we could solve explicitly for c, l, n, y from the FOC .

(b) Competitive equilibrium:

{1} Firm's problem: the firm chooses labor n to maximize its profit treating wage w as given. That is, the firm solves

$$\max_{n} y-wn=\max_{n} n(1-w)$$

The FOC for n implies that $w=\frac{\partial y}{\partial n}=1$. At this equilibrium wage, the consumer is willing to supply $n\in(0,1)$. The consumer would not supply $n=1$ because the marginal utility of leisure would go to infinity $(\lim_{l\to 0} u_2(\cdot,\cdot)=\infty)$. The consumer would not supply $n=1$ because then $y=0$, and thus, $c=0$, and the marginal utility of consumption would go to infinity $(\lim_{c\to 0} u_1(\cdot,\cdot)=\infty)$

{2} Consumer's problem: The consumer maximizes his utility subject to his budget constraint taking his wage w as given. That is, the consumer solves

$$\max_{c,l} u(c,l)+v(\overline{c}) \quad s.t.\; c=w(1-l)$$

Thus, embedding the constraint:

$$\max_{l} u(w(1-l),l)+v(\overline{c})$$

Note that agent takes the average level of consumption as given, i.e., $\overline{c}$ is not a choice variable for an agent:

then, the FOC:

$$-wu_1[w(1-l), l] + u_2[w(1-l), l] = 0$$

{3} Applying the condition that markets for labor and consumption good must clear in the equilibrium (i.e., $w = 1$), we get:

$$-u_1(1-l, l) + u_2(1-l, l) = 0 \tag{2}$$

Obviously, (1) and (2) are not the same, and therefore the allocations will differ in the P.O. and the C.E. cases.

This is because the First Welfare Theorem does not hold when there is an externality (average consumption in the utility function). Any externality distorts the optimal decision made by agents and firms.

(c) in the case of a subsidy the C.E. becomes:

{1} Firm's problem is without any change: $w = 1$, $n \in (0, 1)$

{2} Consumer"s problem: The consumer maximizes his utility subject to his budget constraint taking his wage w and $\bar{c}$ as given. That is, the consumer solves

$$\begin{aligned} & \max_{c,l} u(c, l) + v(\bar{c}) \\ s.t.\ c &= w(1-l) + sc - \tau \Longrightarrow c = \frac{w(1-l) - \tau}{1-s} \end{aligned}$$

or

$$\max_{l} u[\frac{w(1-l) - \tau}{1-s}, l] + v(\bar{c})$$

the FOC now is:

$$-\frac{wu_1(c, l)}{1-s} + u_2(c, l) = 0$$

{3} Government budget constraint holds:

$$\tau = cs$$

{4} Market for consumption good clears:

$$y = c$$

Given the equilibrium conditions {1}-{4}, we have the following equation:

$$-\frac{u_1(c, l)}{1-s} + u_2(c, l) = 0 \tag{3}$$

We can use (3) to solve for c, l, n, y and use $\{1\}$ to get w. To show that the C.E. can be P.O. we need to find a value for s such that (1) and (3)

are the same:

Manipulating (3) we get:

$$u_1(c,l) = (1-s)u_2(c,l) = u_2(c,l) - su_2(c,l) \tag{4}$$

From (1) we have:

$$u_1(c,l) = u_2(c,l) - v'(\overline{c}) \tag{5}$$

Equating the right hand sides of (4) and (5), we get:

$$s = \frac{v'(\overline{c})}{u_2(c,l)} > 0$$

Thus, government has to subsidy the consumer to restore the efficiency. The average consumption $\overline{c}$ acts like a public good, which provides positive utility to every consumer but doesn't charge him any cost, who consequently will consume less private consumption c. The government can correct (internalize) this by providing the appropriate subsidy s to the consumer to induce more private consumption. As the result, the First Welfare Theorem holds and the C.E. becomes the P.O.