AECO 601 **Fall 2021**

Mid-term Exam

Answer the following question. Time allowed: 1 hour 15 min.

1. **Diamond OLG model with preference for leisure**

   We will introduce preference for leisure into the Diamond OLG economy. That means the old can work if they want to.

   Time: discrete, infinite horizon, $t = 1, 2, ...$

   Demography: A mass $N_t = N_0(1+n)^t$ of newborns enter the economy in period t so that n represents the rate of population growth. Everyone lives for 2 periods except for the initial generation of old people who live for one. There is a single price-taking firm.

   Preferences: for the generations born in and after period 1;

$$U(c_{1,t}, l_{1,t}, c_{2,t+1}, l_{2,t+1}) = u(c_{1,t}, l_{1,t}) + \beta u(c_{2,t+1}, l_{2,t+1})$$

   where $c_{i,t} \in \mathbb{R}_+$ is consumption and $l_{i,t} \in [0, 1]$ is leisure in period t and stage i of life. For the initial old generation $\tilde{U}(c_{2,1}, l_{2,1}) = u(c_{2,1}, l_{2,1})$. The instantaneous utility function, $u(.,.)$ is twice differentiable, strictly concave and increasing in both arguments with $\lim_{c\to 0} u_1(c, l) = \infty$, $\lim_{l\to 0} u_2(c, l) = \infty$, $\lim_{c\to\infty} u_1(c, l) = 0$.

   Productive technology: The production function available to firms is $F(K_t, M_t)$ where K_t is the time t capital stock and M_t is the quantity of labor input. $F(.,.)$ is twice differentiable, concave, exhibits constant returns to scale (c.r.s.) and satisfies the Inada conditions.

   Endowments: Everyone has one unit of time in each period of life that can be divided between labor services and leisure. The initial old share an endowment, K_1 of capital. The households equally own the firm. The firm has access to the productive technology.

   Institutions: There are competitive markets every period for the consumption good (numeraire), labor and capital.

   Notation: there is a lot of notation here. One simplification you might want to use is $u_j(i, t) \equiv u_j(c_{i,t}, l_{i,t})$.

   (a) Write down the Social Planner's problem and obtain the first order conditions. (Assume that the Planner weights each generation equally.) You may find it easier to write it out on a per young person basis so that k_t and m_t are the per young person capital stock and labor respectively.

   The Planner solves

$$\max_{\{c_{1,t}, l_{1,t}, c_{2,t}, l_{1,t}, K_t, M_t\}} u(c_{2,1}, l_{2,1}) + \sum_{t=1}^{\infty} u(c_{1,t}, l_{1,t}) + \beta u(c_{2,t+1}, l_{2,t+1})$$

$$\text{s.t.} \quad N_t c_{1,t} + N_{t-1} c_{2,t} + K_{t+1} = F(K_t, M_t)$$

$$M_t = N_t(1 - l_{1,t}) + N_{t-1}(1 - l_{2,t})$$

Writing out the Lagrangian (in terms of per young person) we get

$$\begin{aligned}
\mathcal{L} &= u(c_{2,1}, l_{2,1}) + \sum_{t=1}^{\infty} u(c_{1,t}, l_{1,t}) + \beta u(c_{2,t+1}, l_{2,t+1}) \\
&\quad + \lambda_t \left[c_{1,t} + \frac{c_{2,t}}{1+n} + (1+n)k_{t+1} - F(k_t, m_t) \right] \\
&\quad + \mu_t \left[m_t - (1 - l_{1,t}) - \frac{1 - l_{2,t}}{1+n} \right]
\end{aligned}$$

The FOCs are

$$\begin{aligned}
c_{2,1} &: \quad u_1(2,1) + \frac{\lambda_1}{1+n} = 0 \\
l_{2,1} &: \quad u_2(2,1) + \frac{\mu_1}{1+n} = 0 \\
c_{1,t} &: \quad u_1(1,t) + \lambda_t = 0 \\
l_{1,t} &: \quad u_2(1,t) + \mu_t = 0 \\
c_{2,t} &: \quad \beta u_1(2,t) + \frac{\lambda_t}{1+n} = 0 \\
l_{2,t} &: \quad \beta u_2(2,t) + \frac{\mu_t}{1+n} = 0 \\
k_t &: \quad (1+n)\,\lambda_{t-1} - \lambda_t F_1(k_t, m_t) = 0 \\
m_t &: \quad \mu_t - \lambda_t F_2(k_t, m_t) = 0
\end{aligned}$$

(b) Write down and obtain first order conditions for the problem faced by the individuals born in period t.

The individual solves

$$\begin{aligned}
\max_{\{c_{1,t}, l_{1,t}, c_{2,t}, l_{1,t}, s_{t+1}\}} \quad & u(c_{1,t}, l_{1,t}) + \beta u(c_{2,t+1}, l_{2,t+1}) \\
\text{s.t.} \quad & (1 - l_{1,t})w_t = c_{1,t} + s_{t+1} \\
& R_{t+1}s_{t+1} + (1 - l_{2,t})w_{t+1} = c_{2,t+1}
\end{aligned}$$

After substituting $c_{1,t}$ and $c_{2,t+1}$ out of problem using the constraints the FOCs become:

$$\begin{aligned}
l_{1,t} &: \quad -w_t u_1(1,t) + u_2(1,t) = 0 \\
l_{2,t+1} &: \quad -w_{t+1} u_1(2,t+1) + u_2(2,t+1) = 0 \\
s_{t+1} &: \quad -u_1(1,t) + R_{t+1}\beta u_1(2,t+1) = 0
\end{aligned}$$

(c) Write down and solve the problem of the single, price taking firm.

The firm solves

$$\max_{K_t, M_t} F(K_t, M_t) - w_t M_t - R_t K_t$$

FOCs are

$$\begin{aligned}
F_1(K_t, M_t) &= R_t \\
F_2(K_t, M_t) &= w_t
\end{aligned}$$

(d) Write down the market clearing conditions for labor, capital and goods.

The market clearing conditions are

$$
\begin{aligned}
\text{Labor} &: \quad M_t = N_t(1-l_{1,t}) + N_{t-1}(1-l_{2,t}) \\
\text{Capital} &: \quad K_t = N_{t-1}s_t \\
\text{Goods} &: \quad N_t c_{1,t} + N_{t-1}c_{2,t} + K_{t+1} = F(K_t, M_t)
\end{aligned}
$$

Alternatively in per young person terms:

$$
\begin{aligned}
\text{Labor:} &\qquad m_t = 1 - l_{1,t} + \frac{1-l_{2,t}}{1+n} \\
\text{Capital:} &\qquad k_t = \frac{s_t}{1+n} \\
\text{Goods:} &\qquad c_{1,t} + \frac{c_{2,t}}{1+n} + (1+n)k_{t+1} = F(k_t, m_t)
\end{aligned}
$$

(e) Define a competitive equilibrium.

Definition 1 *A competitive equilibrium is an allocation,* $\{c_{1,t}, l_{1,t}, c_{2,t}, l_{1,t}, K_t, M_t\}_{t=1}^{\infty}$ *and a sequence of prices,* $\{R_t, w_t\}_{t=1}^{\infty}$ *such that*

i) Given prices, the allocation solves the individuals' and firm's problems.

ii) Markets clear.

(f) Eliminate prices to obtain a characterization of the steady state competitive equilibrium in per young person terms. (Express steady state variables by dropping the time subscript)

$$
\begin{aligned}
\frac{u_2(c_1, l_1)}{u_1(c_1, l_1)} &= F_2(k, m) \\
\frac{u_2(c_2, l_2)}{u_1(c_2, l_2)} &= F_2(k, m) \\
\frac{u_1(c_1, l_1)}{u_1(c_2, l_2)} &= \beta F_1(k, m) \\
m &= 1 - l_1 + \frac{1-l_2}{1+n} \\
(1+n)k &= s
\end{aligned}
$$

where the variables without time subscripts represent steady state values. It looks like there are not enough equations but we can think of the consumption values as being substituted out using the individual budget constraints:

$$
(1-l_1)F_2(k,m) = c_1 + s
$$
$$
sF_1(k,m) + (1-l_2)F_2(k,m) = c_2
$$

(g) Obtain a per young person characterization of the steady state Planner's solution.

$$
\begin{aligned}
\frac{u_1(c_1,l_1)}{u_1(c_2,l_2)} &= \beta(1+n) \\
\frac{u_2(c_1,l_1)}{u_2(c_2,l_2)} &= \beta(1+n) \\
F_1(k,m) &= 1+n \\
\lambda &= \mu F_2(k,m) \\
\frac{u_1(c_1,l_1)}{u_2(c_1,l_1)} &= \frac{\lambda}{\mu} \\
\frac{u_1(c_2,l_2)}{u_2(c_2,l_2)} &= \frac{\lambda}{\mu}
\end{aligned}
$$

(h) Show that the allocations coincide when $F_1(k,m) = 1+n$ where k and m are the steady state per young person capital stock and labor demand respectively. Briefly explain the result.

Setting $F_1(k,m) = 1+n$ in the laissez-faire economy and eliminating the Lagrange multipliers from the Planner's solution clearly yields the same set of equations. Allowing for a labor-leisure choice does not eliminate the potential for inefficiency in the competitive equilibrium. That is driven by the inability to trade with the dead and unborn.