Macroeconomics I: Mid-Term Exam

Infinite horizon model with government facilitated production

Time: Discrete, infinite horizon

Demography: A single representative infinite lived individual/household

Preferences: Discounted lifetime utility, U, given by

$$U = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(l_t)\right]$$

where $c_t \in \boldsymbol{R}_+$ is period t consumption, $l_t \in [0,1]$ is leisure and $\beta \in [0,1)$ is a constant discount factor. The felicity functions $u(.)$ and $v(.)$ are strictly increasing, strictly concave and exhibit infinite marginal utility at 0.

Technology: Per capita output y_t given by

$$y_t = z_t f(g_t) n_t$$

where $z_t = z_0(1+\gamma)^t$ is the total factor productivity (TFP), g_t is government spending and n_t is labor input. The growth rate of TFP, $\gamma > 0$. The function $f(.)$ is strictly increasing and strictly concave and represents the role of government regulation in improving the effectiveness of industry.

Endowments: The household has one unit of time to devote to either labor supply or leisure. The household owns the firms who produce the consumption good. Firms have access to the technology.

Institutions: There is a government with power to levy taxes, τ_t, on the household. Government spending, g_t, goes entirely into improving the firms' productivity as indicated above. In the market economy there are markets each period for the consumption good and labor. (No government bonds or inside money.)

(a) Write down the household's full problem and reduce it to one of choosing l_t for each t. Obtain the first-order condition for this problem.

The household solves

$$\max_{\{c_t,l_t\}} \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(l_t)\right]$$
$$\text{s.t.} \quad c_t = w_t(1-l_t) - \tau_t$$

or

$$\max_{\{l_t\}} \sum_{t=0}^{\infty} \beta^t \left[u(w_t(1-l_t) - \tau_t) + v(l_t)\right]$$

FOC:

$$-w_t u'(w_t(1-l_t) - \tau_t) + v'(l_t) = 0$$

(b) Write down and solve the firms' problem.

Firms solve:

$$\max_{n_t}\{z_t f(g_t) n_t - w_t n_t\}$$

which implies $w_t = z_t g_t$.

(c) Write down the government budget constraint, the market clearing constraints for consumption goods and labor (assume there is 1 firm) and define a competitive equilibrium.

$$\begin{aligned} \text{GBC} &: \quad g_t = \tau_t \\ \text{Goods} &: \quad c_t + g_t = z_t f(g_t) n_t \\ \text{Labor} &: \quad n_t = 1 - l_t \end{aligned}$$

Definition: A competitive equilibrium is an allocation, $\{c_t, l_t, n_t\}$ and a sequence of wages $\{w_t\}$ such that given prices the allocation solves the household's and the firm's problem, markets clear and the government budget constraint holds.

(d) Obtain an equation that characterizes equilibrium leisure, l_t^* as a (implicit) function of z_t and g_t.

$$z_t f(g_t) u'(z_t f(g_t)(1 - l_t^*) - g_t) - v'(l_t^*) = 0 \tag{1}$$

(e) Assuming the government is motivated by interests of the household, write down the government's problem and obtain the first order condition for each t.

Government solves:

$$\max_{\{g_t\}} \sum_{t=0}^{\infty} \beta^t \left[u(z_t f(g_t)(1 - l_t^*) - g_t) + v(l_t^*)\right]$$

FOC:

$$\left(z_t f'(g_t)(1 - l_t^*) - 1\right) u'(c_t) - \left[z_t f(g_t) u'(z_t f(g_t)(1 - l_t^*) - g_t) - v'(l_t^*)\right] \frac{dl_t^*}{dg_t} = 0$$

the contents of the square brackets are zero by by equation (1) (it is the envelope theorem) so

$$z_t f'(g_t)(1 - l_t^*) = 1 \tag{2}$$

(f) Write down the Planner's problem for this economy. Does His solution differ from what emerges from the decentralized economy with the benevolent government? Why/why not?

Planner solves

$$\max_{\{c_t, l_t, g_t\}} \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + v(l_t)\right]$$
$$\text{s.t} \qquad c_t = z_t g_t (1 - l_t) - g_t$$

Sub in for c_t. FOC's for g_t and l_t are identical to (1) and (2). There are no distortions and the government has the same objective function as the Planner.

Now assume $u(c) = \ln(c)$ and $v(l) = \ln(l)$ where ln(.) is the natural logarithm function. Also assume $f(g_t) = g_t^{\frac{1}{2}}$.

(g) Solve for l_t^*, optimal g_t and c_t. What is the share of output going to household and government spending?

Equations (1) and (2) become

$$\begin{aligned} \frac{z_t g_t^{\frac{1}{2}}}{z_t g_t^{\frac{1}{2}}(1 - l_t^*)} &= \frac{1}{l_t^*} \\ \frac{z_t(1 - l_t^*)}{2 g_t^{\frac{1}{2}}} &= 1 \end{aligned}$$

So

$$g_t = \frac{z_t^2 (1 - l_t^*)^2}{4}$$

Into first equation gives $l_t^* = \dfrac{1}{3}$ for all t and $g_t = \dfrac{z_t^2}{9} = \dfrac{z_0^2\,(1+\gamma)^{2t}}{9}$. Output is $z_t f(g_t)(1 - l_t^*) = \dfrac{2 z_0^2\,(1+\gamma)^{2t}}{9}$. So $c_t = g_t$ and both represent half of output each.

(h) If γ is close to zero what is the approximate (net) growth rate of the economy?

The gross growth rate is $(1+\gamma)^2 = 1 + 2\gamma + \gamma^2$. Net growth rate is approximately 2γ.