1. **Optimal growth with an investment function**

   Time: Discrete; infinite horizon

   Demography: A single representative (price taking) consumer/producer household and a single representative (price taking) firm.

   Preferences: the instantaneous household utility function over, individual consumption, c, is $u(c)$. The function $u(.)$ is twice differentiable, strictly increasing and strictly concave and bounded above with marginal utility going to infinity as c goes to zero. The discount factor is $\beta \in (0,1)$.

   Technologies: *Productive*: There is a constant returns to scale aggregate technology over capital and labor such that output per unit of labor employed is $f(k)$, where k is capital input per unit of labor; $f(.)$ is twice differentiable, strictly increasing and concave with $f(0) = 0$, $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$. Capital depreciates at the rate $\delta < 1$.

   Investment: Goods used for investment, i_t, do not convert one-for-one with capital. Instead there is an investment function $g(i_t) = k_{t+1} - (1-\delta)k_t$. Where $g(.)$ is twice differentiable, strictly increasing, bounded above and strictly concave with $g'(0) = 1$. For notational simplicity use $\gamma(.)$ as the inverse function of $g(.)$ (i.e. $i_t = \gamma(k_{t+1} - (1-\delta)k_t)$). Also for notational brevity you might want to use $\hat{\imath}_t = k_{t+1} - (1-\delta)k_t$ so $i_t = \gamma(\hat{\imath}_t)$.

   Endowments: The household's initial capital stock is k_0, it also has 1 unit of labor each period. The firm has access to the technology, $f(.)$. The household has access to the investment technology, $g(.)$.

   (a) Write down and solve the Planner's problem for this economy.

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\} \\ \text{s.t.} \quad c_t &= f(k_t) - \gamma(\hat{\imath}_t) \\ \text{where} \quad \hat{\imath}_t &= k_{t+1} - (1-\delta)k_t \\ &\quad \text{and } k_0 \text{ given.} \end{aligned}$$

   After substitution the FOC for k_{t+1} is

$$-u'(c_t)\gamma'(\hat{\imath}_t) + \beta V'(k_{t+1}) = 0.$$

   The envelope equation is

$$V'(k_t) = u'(c_t)\left[f'(k_t) + \gamma'(\hat{\imath}_t)(1-\delta)\right].$$

   Rolling this forward and substituting into the FOC yields the Euler equation

$$u'(c_t)\gamma'(k_{t+1} - (1-\delta)k_t) = \beta u'(c_{t+1})\left[f'(k_{t+1}) + \gamma'(k_{t+2} - (1-\delta)k_{t+1})(1-\delta)\right].$$

   This along with the resource constraint characterize the solution to the Planner's problem.

(b) Briefly explain the Planner's Euler equation that you obtained

At the optimum, the marginal cost of giving up a unit consumption should be equal to the marginal benefit of doing so. Consuming one fewer unit costs the marginal utility from consumption and is converted into capital at the marginal cost $\gamma'(\hat{\imath}_t)$. That is, it yields $1/\gamma'(\hat{\imath}_t)$ units of capital. Each additional amount of capital generates $f'(k_{t+1})$ of output and leaves $1-\delta$ units of undepreciated capital which can be used to reduce the period $t+1$ investment by $\gamma'(\hat{\imath}_{t+1})$. That extra consumption in period $t+1$ yields $u'(c_{t+1})$ utils but has to be discounted by β from the perspective from period t.

(c) Write down and solve the household's problem.

$$
\begin{aligned}
v(k_t) &= \max_{k_{t+1}} \{u(c_t) + \beta v(k_{t+1})\} \\
\text{s.t.} \quad c_t &= r_t k_t + w_t - \gamma(\hat{\imath}_t) \\
\text{where} \quad \hat{\imath}_t &= k_{t+1} - (1-\delta)k_t \\
& \quad \text{and } k_0, \{r_t\}, \{w_t\} \text{ given}
\end{aligned}
$$

The FOC is

$$-u'(c_t)\gamma'(\hat{\imath}_t) + \beta v'(k_{t+1}) = 0.$$

The envelope condition is

$$v'(k_t) = u'(c_t)\left[r_t + \gamma'(\hat{\imath}_t)(1-\delta)\right].$$

So the Household's Euler equation is

$$u'(c_t)\gamma'(\hat{\imath}_t) = \beta u'(c_{t+1})\left[r_{t+1} + \gamma'(\hat{\imath}_{t+1})(1-\delta)\right].$$

(d) Write down and solve the firm's problem.

Every period the firm solves

$$
\begin{aligned}
&\max_{k_t^f} \left\{ f(k_t^f) - r_t k_t^f - w_t \right\} \\
&\{r_t\}, \{w_t\} \text{ given}
\end{aligned}
$$

The only solution consistent with an interior outcome is $r_t = f'(k_t^f)$. Then constant returns to scale implies, $w_t = f(k_t^f) - f'(k_t^f)k_t^f$.

(e) Write out the market clearing conditions for capital and consumption goods and, define a competitive equilibrium.

Market clearing requires

$$
\begin{aligned}
\text{Capital} &: \quad k_t^f = k_t \\
\text{Goods} &: \quad c_t = f(k_t) - \gamma(k_{t+1} - (1-\delta)k_t)
\end{aligned}
$$

Definition 1 *A competitive equilibrium is an allocation,* $\left\{c_t, k_t, k_t^f\right\}$ *and prices,* $\{r_t w_t\}$, *such that*

- *given prices, the allocation solves the household's and firm's problems*
- *markets clear*

(f) Solve for the characterization of equilibrium in terms of c_t and k_t. Is it the same as the Planner's solution? Briefly explain your answer.

After substitution the equations that characterize equilibrium are

$$\begin{aligned} c_t &= f(k_t) - \gamma(k_{t+1} - (1-\delta)k_t) \\ u'(c_t)\gamma'(k_{t+1} - (1-\delta)k_t) &= \beta u'(c_{t+1}) \left[f'(k_{t+1}) + \gamma'(k_{t+2} - (1-\delta)k_{t+1})(1-\delta) \right] \end{aligned}$$

They are identical to the Planner's solution. Efficiency of the competitive equilibrium is not affected by the non-linearity of the way consumption goods can be turned into capital. It is purely technological and there are no externalities introduced.

2. **One-sided search with 2 job separation rates.**

Time: Discrete, infinite horizon.

Demography: A unit mass continuum of workers.

Preferences: All are risk-neutral (i.e. $u(x) = x$) and discount the future at the rate r.

Endowments:

When *unemployed,* each worker receives utility b per period from leisure.

Also each period they are unemployed, with probability α, workers receive a job offer $w \sim F$ on $[0, \bar{w}]$. The distribution function, $F(.)$ has continuous density, $f(.)$.

In addition to the heterogeneity coming from wages, jobs also differ in their separation probabilities. A proportion η of offers are for jobs that will make the worker unemployed again with probability λ_H and a proportion $1-\eta$ of offers are for jobs that will make the worker unemployed again with probability λ_L where $\lambda_H > \lambda_L$. The job separation rate is statistically independent of the wage realization. Workers are made aware of the separation rate of the job when they receive the offer.

So, when *employed* each period workers in a type $J \in \{H, L\}$ job receive their wage, w, but with probability λ_J lose their jobs to become unemployed.

(a) Use V_u to represent the value to unemployment. Use $V_J(w)$ to represent the value to employment at wage w in a type $J = H, L$ job. Write down the flow asset value equations and briefly explain each one.

$$\begin{aligned} rV_J(w) &= w + \lambda_J \left(V_u - V_J(w) \right), \quad J = H, L \\ rV_u &= b + \alpha\eta \mathbb{E}_w \left[\max \left\{ V_H(w) - V_u, 0 \right\} \right] + \alpha(1-\eta) \mathbb{E}_w \left[\max \left\{ V_L(w) - V_u, 0 \right\} \right] \end{aligned}$$

(b) Define w_J^*, as the reservation wage associated with accepting jobs of type $J = H, L$ and show that $w_H^* = w_L^* (\equiv w^*)$. Explain this result.

From the first value function we get

$$V_J(w) = \frac{w + \lambda_J V_u}{r + \lambda_J} \quad J = H, L \tag{1}$$

Because by definition, $V_J(w_J^*) = V_u$, evaluating either of $V_J(w)$ at w_J^* yields

$$V_u = \frac{w_J^* + \lambda_J V_u}{r + \lambda_J} \quad \text{so } rV_u = w_J^* \quad J = H, L. \tag{2}$$

At the reservation wage the worker is indifferent between taking the job and rejecting it so the propensity to get laid-off does not enter the choice of the reservation wage.

(c) Solve for the reservation wage equation that contains only w^* and model parameters.

From equations (1) we obtain,

$$V_J(w) - V_u = \frac{w + \lambda_J V_u}{r + \lambda_J} - V_u = \frac{w - rV_u}{r + \lambda_J} = \frac{w - w^*}{r + \lambda_J}$$

Then substituting into the V_u value function we obtain,

$$w^* = b + \alpha \left[\frac{\eta \left(r + \lambda_L\right) + \left(1 - \eta\right) \left(r + \lambda_H\right)}{\left(r + \lambda_L\right) \left(r + \lambda_H\right)} \right] \int_{w^*}^{\bar{w}} \left(w - w^*\right) dF(w)$$

(d) Obtain $\frac{dw^*}{d\eta}$. Explain how to reconcile this result with that in part b.

$$\frac{dw^*}{d\eta} = \frac{-\left[-\alpha \frac{(r+\lambda_L)-(r+\lambda_H)}{(r+\lambda_L)(r+\lambda_H)}\right] \int_{w^*}^{\bar{w}} \left(w - w^*\right) dF(w)}{-\alpha \left[\frac{\eta(r+\lambda_L)+(1-\eta)(r+\lambda_H)}{(r+\lambda_L)(r+\lambda_H)}\right] \left[-(1 - F(w^*)\right]}$$

the sign of which boils down to that of $\lambda_L - \lambda_H < 0$.

So even though the reservation wage is invariant to an individual's separation rate, the proportion of high versus low separation rate jobs has an impact on it. This is because the marginal job has the same value but the average job within each type does not. The workers in general prefer the low separation rate jobs. Increasing η makes them generally worse off so they lower their reservation wage.