AECO601 Fall 2020

**Final Exam:** Answer both questions time allowed 3 hours. You may consult your notes or text books but you may not communicate with anyone about the exam. You must email your answers to me as a pdf by 11:30am US Eastern time. Please make sure that the files you upload are legible.

1. **Optimal growth with government spending preferences**

   **Time:** Discrete; infinite horizon

   **Demography:** A single representative (price taking) consumer/producer household and a single representative (price taking) firm.

   **Preferences:** the instantaneous household utility function over, individual consumption, $c$, and government spending, $g$, is $U(c, g) = u(c) + v(g)$. Both $u(.)$ and $v(.)$ are twice differentiable, strictly increasing and strictly concave and bounded above with marginal utility going to infinity as $c$ or $g$ go to zero. The discount factor is $\beta \in (0, 1)$.

   **Productive Technology:** There is a constant returns to scale aggregate technology over capital and labor such that output per unit of labor employed is $f(k)$, where $k$ is capital input per unit of labor; $f(.)$ is twice differentiable, strictly increasing and concave with $f(0) = 0$, $\lim_{k\to0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$. Capital depreciates at the rate $\delta < 1$.

   **Endowments:** The households' initial capital stock is $k_0$, it also has 1 unit of labor each period. Firms have access to the technology, $f(.)$.

   (a) The Social Planner wants to choose paths for consumption **and government spending** to maximize welfare. Write down the Planner's problem in recursive form (i.e. using value functions) and solve it.

   The Planner's problem is

$$\begin{aligned} V(k_t) &= \max_{c_t, g_t, k_{t+1}} \{u(c_t) + v(g_t) + \beta V(k_{t+1})\} \\ \text{s.t.} \quad f(k_t) + (1-\delta)k_t &= k_{t+1} + c_t + g_t \end{aligned}$$

   Substitute $c_t$ out and FOC's become:

$$\begin{aligned} k_{t+1} &: \quad -u'(c_t) + \beta V'(k_{t+1}) = 0 \\ g_t &: \quad -u'(c_t) + v'(g_t) = 0 \end{aligned}$$

   Envelope condition:

$$V'(k_t) = u'(c_t)\left[f'(k_t) + (1-\delta)\right]$$

   These imply

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1})\left[f'(k_{t+1}) + (1-\delta)\right] \\ u'(c_t) &= v'(g_t) \end{aligned}$$

   **Market Economy Institutions:** In the market economy there is a government that has to meet an exogenous (for now) stream a of expenditures $\{g_t\}$. To do so it imposes a sequence of proportional tax rates $\{\tau_t\}$. The whole of the household's current period income (from capital and labor) is taxed proportionally each period.

(b) Write down and solve the household's problem

Household's problem:

$$\begin{aligned} V(k_t) &= \max_{c_t, g_t, k_{t+1}} \{u(c_t) + v(g_t) + \beta V(k_{t+1})\} \\ \text{s.t.} \quad c_t &= (w_t + r_t k_t)(1-\tau_t) + (1-\delta)k_t - k_{t+1} \end{aligned}$$

After substitution we have

$$k_{t+1}: \quad -u'(c_t) + \beta V'(k_{t+1}) = 0$$

the envelope condition is

$$V'(k_t) = u'(c_t)\left[r_t(1-\tau_t) + (1-\delta)\right]$$

so

$$u'(c_t) = \beta u'(c_{t+1})\left[r_{t+1}(1-\tau_{t+1}) + (1-\delta)\right]$$

(c) Write down and solve the firm's problem

The firm solves

$$\max_{k_t^f} f(k_t^f) - r_t k_t - w_t$$

which leads to

$$\begin{aligned} f'(k_t^f) &= r_t \\ f(k_t^f) - k_t^f f'(k_t^f) &= w_t \end{aligned}$$

(d) Write down the market clearing conditions and the government budget constraint. Define a competitive equilibrium.

Market clearing:

$$\begin{aligned} \text{Goods} &: \quad f(k_t) + (1-\delta)k_t = k_{t+1} + c_t + g_t \\ \text{Capital} &: \quad k_t^f = k_t \end{aligned}$$

Government budget constraint (GBC)

$$g_t = (w_t + r_t k_t)\tau_t = \tau_t f(k_t).$$

A balanced budget competitive equilibrium is an allocation, $\left\{k_{t,} c_t, k_t^f\right\}$, prices $\{r_t, w_t\}$ and taxes $\{\tau_t\}$ such that (i) given taxes and prices the allocation solves the firms' and households problems, (ii) markets clear and, (iii) GBC holds.

(e) Obtain a system of 2 difference equations that characterizes equilibrium in terms of $c_t$ and $k_t$ for given $\{g_t\}$.

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1})\left[f'(k_{t+1})\left(1 - \frac{g_{t+1}}{f(k_{t+1})}\right) + (1-\delta)\right] \\ f(k_t) + (1-\delta)k_t &= k_{t+1} + c_t + g_t \end{aligned}$$

(f) Is the competitive equilibrium efficient for given $g_t$? Explain your answer

No, the household does not have the right incentives to consume and save because of the proportional tax.

(g) Write down the system of equations that characterize a **steady state** equilibrium $(k^*, c^*)$ when $g_t = g$ for all $t$.

$$\begin{aligned} \frac{1}{\beta} &= \left[ f'(k^*) \left(1 - \frac{g}{f(k^*)}\right) + (1-\delta) \right] \\ c &= f(k^*) - \delta k^* - g \end{aligned}$$

(h) Now we consider a situation in which the government can control $g$ but does so using $\tau$ a constant tax rate. Obtain a system of equations in $k^*$ and $\tau$ that characterizes the maximum level of government spending, $g$, achievable in steady state.

The problem is

$$\begin{aligned} & \max_{\tau} \tau f(k^*) \\ \text{s.t.} \quad 1 &= \beta \left( f'(k^*)(1-\tau) + (1-\delta) \right) \end{aligned}$$

Using the constraint we get

$$\frac{dk^*}{d\tau} = \frac{f'(k^*)}{f''(k^*)(1-\tau)}$$

So the first order condition is

$$f(k) + \tau f'(k) \frac{dk^*}{d\tau} = f(k) + \frac{\tau f'^2(k^*)}{f''(k^*)(1-\tau)} = 0$$

The system is this plus the constraint

$$1 = \beta \left( f'(k^*)(1-\tau) + (1-\delta) \right)$$

(i) Now let $f(k) = k^\alpha$ and obtain the government spending maximizing tax rate, $\hat{\tau}$. Explain why this exists.

When we substitute in we get $\tau = 1 - \alpha$. There is a Laffer type curve so that revenue is low with low taxes and low again if taxes get too high because the household does not save enough to sustain high output.

2. **Diamond Coconut Economy with random nut quality**

**Time:** Discrete, infinite horizon

**Geography:** A trading island and a production island.

**Demography:** A mass 1 of ex ante identical individuals with infinite lives.

**Preferences:** The common discount rate is $r$. Consumption of one's own produce yields 0 utils. Consumption of someone's output of quality $u \in [0, \bar{u}]$ yields $u$ utils.

**Note:** the value of $u$ is not match specific, everyone agrees on the quality of a given nut which is also known to its producer.

**Productive Technology:** On the production island individuals come across a tree with a coconut with probability $\alpha$ each period. Once they find a tree obtaining a coconut is costless (i.e. there is no distribution of tree climbing costs and $c = 0$). The quality of the nut they find is continuously distributed $F(u)$ on $[0, \bar{u}]$. The producer observes the quality of the nut and decides whether to pick it up and carry it to the trading island or whether to ignore that nut and search for a better one. (The reason you might not pick up a nut is that if it is low quality, people on the trading island may not give up their nut for yours. Then you would remain on the trading island forever.)

**Matching Technology:** On the trading island people with coconuts meet each other with probability $\gamma$ each period; $\gamma$ is a constant (i.e. invariant to the number of individuals on the trading island). When you meet someone, you both decide whether to trade based on the quality of the nuts you are both carrying.

**Navigation:** Travel between islands is free and instantaneous.

**Endowments:** Everyone has a boat and starts off with one of their own coconuts of quality $\bar{u}$. (This is just to set the economy going we will focus on steady state equilibrium.)

(a) Suppose that everyone else has a reservation nut quality, $u^*$. This means that on the production island no one will pick up a nut of quality less than $u^*$ and on the trading island, they will not give up their nut unless the one you offer them is at least of quality $u^*$. On that basis write down the flow value equations for the individual who has to decide for him/herself which nuts to accept on the trading island.

$$\begin{aligned} rV_T &= \gamma \boldsymbol{E}_{u \geq u^*} \left[ \max \left\{ V_P + u - V_T, 0 \right\} \right] \\ rV_P &= \alpha(1 - F(u^*))(V_T - V_P) \end{aligned}$$

(b) Given $u^*$, obtain an implicit equation for $\hat{u}$ the reservation quality accepted by the individual in exchange for the nut s/he is carrying.

So,

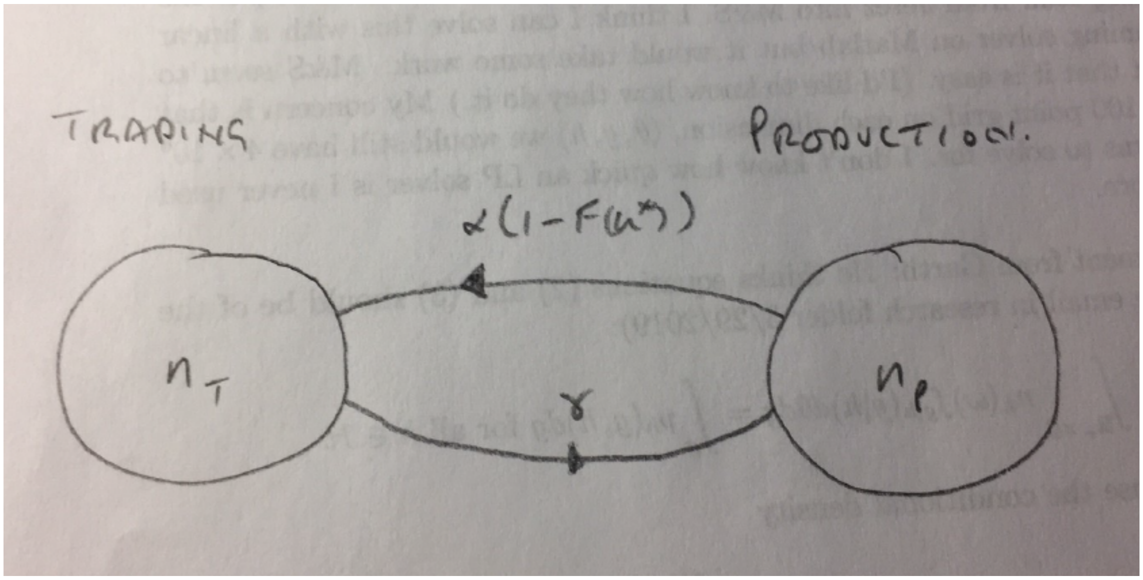
$$rV_T = \gamma \int_{\max\{\hat{u}.u^*\}}^{\bar{u}} \frac{(V_P + u - V_T)\, dF(u)}{1 - F(u^*)}$$

where $\hat{u} = V_T - V_P$. Differencing the $V_P$ and $V_T$ equations yields

$$\left[ r + \alpha(1 - F(u^*)) \right] \hat{u} = \frac{\gamma}{1 - F(u^*)} \int_{\max\{\hat{u}.u^*\}} (u - \hat{u})\, dF(u)$$

(c) Define a steady state search equilibrium in which everyone on the trading island accepts all coconuts. (**Hint:** in equilibrium $u^* = \hat{u}$.)

A steady-state search equilibrium is pattern of trade, characterized by a common reservation coconut quality, $u^*$, such that when everyone else conforms to that pattern of trade no one else wants to deviate from it.




(d) Obtain a characterization of equilibrium.

We need to impose that $\hat{u} = u^*$. So that,

$$[r + \alpha(1 - F(u^*))]\, u^* = \frac{\gamma}{1 - F(u^*)} \int_{u^*}^{\bar{u}} (u - u^*)\, dF(u)$$

(e) Draw a diagram showing the flow rates between and populations on the islands. Use this to obtain an expression in terms of equilibrium $u^*$ for the steady state population, $n_T$ on the trading island.

In steady state

$$\alpha(1 - F(u^*))(1 - n_T) = \gamma n_T$$

so

$$n_T = \frac{\alpha(1 - F(u^*))}{\alpha(1 - F(u^*)) + \gamma}$$

(f) Now let $F(u)$ be uniform on $[0, 1]$ so that

$$F(u) = \begin{cases} 0 \text{ for } u < 0 \\ u \text{ for } u \in [0, 1] \\ 1 \text{ for } u > 1 \end{cases} .$$

Use the characterization of equilibrium to obtain an implicit function for $u^*$.

$$\begin{aligned}
[r + \alpha(1 - u^*)]\,(1 - u^*)u^* &= \gamma \int_{u^*} (u - u^*)\, du \\
[r + \alpha(1 - u^*)]\,(1 - u^*)u^* &= \frac{1}{2}\gamma(1 - u^*)^2 \\
2\,[r + \alpha(1 - u^*)]\, u^* - \gamma(1 - u^*) &= 0
\end{aligned}$$

(g) Now simplify the problem further by setting $r = 0$. Obtain the possible values of $u^*$. Explain the result.

If $r = 0$ we obtain

$$2\alpha(1 - u^*)u^* - \gamma(1 - u^*) = 0$$

which has 2 roots, $u^* = 1$ and $u^* = \frac{\gamma}{2\alpha}$. There are nominally 2 possible equilibria each associated with one root. The first root tells us that when everyone expects anyone on the trading island to not take nuts of any quality then no one will pick any up. There is no one on the trading island and so the belief that they will not accept any nuts is justified. This is like a trivial equilibrium. For the second root, as long as $\gamma < 2\alpha$ there is an interior equilibrium. If $\gamma > 2\alpha$, this equilibrium also reverts to the trivial one and no consumption occurs. This is an unraveling equilibrium similar to a lemons market. If those in the trading island meet fast enough they are very picky, always raising their threshold of acceptability. This happens because when they make their choice as to whether to accept a nut of a given quality or not, they do not take into account the impact on producers in terms of how hard such nuts are to find. When $\gamma > 2\alpha$ there may be other non-trivial equilibria that we have not considered. For example there could be stratification of goods into classes on the trading island as in Burdett and Coles (*QJE* 1997).