Final Exam: Answer both questions time allowed 2 hours

1. **Degenerate Diamond coconut economy with 2 qualities of nut**

   Time: Discrete, infinite horizon

   Geography: A trading island and a production island.

   Demography: A mass of 1 of ex ante identical individuals with infinite lives.

   Preferences: The common discount rate is r. Consumption of ones own produce yields 0 utils. Consumption of someone's yields either u or x ($u > x > 0$) utils.

   Productive Technology: On the production island individuals come across a tree with a coconut with probability α each period. Once they find a coconut tree obtaining a nut is costless (i.e. there is no distribution of tree climbing costs and $c = 0$). With probability $\phi > 0$, the the nut they find is of high quality in that the eventual consumer will get u utils from it's consumption. With probability $1 - \phi$ the nut is of low quality so that the eventual consumer just gets x utils.

   Matching Technology: On the trading island people with coconuts meet each other with probability γ each period; γ is a constant (i.e. invariant to the number of individuals on the trading island).

   Navigation: Travel between islands is free and instantaneous.

   Endowments: Everyone has a boat and starts off with one of their own coconuts.

   Steady state equilibria: We will explore the possibility that there are multiple equilibria of the model. In a Type 1 equilibrium everyone accepts both types of nut on the trading island and therefore everyone picks up both types of nut on the production island. In a Type 2 equilibrium individuals on the trading island reject low quality nuts which means no one picks them up on the production island.

   (a) Write down the flow asset value equations relevant for a Type 1 equilibrium. Use V_T and V_P for the values of being on the trading and production islands respectively. (Note that, in this equilibrium, because any nut that I bring to the trading island will be accepted for whatever nut my trading partner has, V_T does not depend on the quality of the nut I am carrying.)

   $$\begin{aligned} rV_P &= \alpha(V_T - V_P) \\ rV_T &= \gamma\left[\phi(u + V_P - V_T) + (1-\phi)(x + V_P - V_T)\right] \end{aligned}$$

   (b) Define a steady-state search equilibrium. (This should be good for both types of equilibrium.)

   Definition: A steady-state search equilibrium is a stationary pattern of trade such that when all other individuals in the conform to it, no one wishes to deviate

(c) Solve for $V_T - V_P$. (You do not need to obtain the individual values of V_T and V_P.)

From differencing the flow asset value equations we get

$$V_T - V_P = \frac{\gamma\left[\phi u + (1-\phi)x\right]}{r+\alpha+\gamma}$$

(d) Solve for the value of $\frac{u}{x}$ in terms of the other parameters below which all individuals conform to the equilibrium (i.e. accept low quality nuts)? Hint: when would they prefer to wait for a high quality nut rather than take every one that comes along?

For them to prefer to take all nuts rather than wait for a high quality nut we need

$$V_P + x > V_T.$$

That is $V_T - V_P < x$. This implies

$$\frac{u}{x} < \frac{r+\alpha+\phi\gamma}{\phi\gamma}$$

(e) Write down the flow asset value equations relevant for a Type 2 equilibrium.

$$\begin{aligned} rV_P &= \alpha\phi(V_T - V_P) \\ rV_T &= \gamma\left[u + V_P - V_T\right] \end{aligned}$$

(f) Solve for $V_T - V_P$. (You do not need to obtain the individual values of V_T and V_P.)

$$V_T - V_P = \frac{\gamma u}{r+\alpha\phi+\gamma}$$

(g) Solve for the value of $\frac{u}{x}$ in terms of the other parameters above which all individuals conform to the equilibrium (i.e. reject low quality nuts)?

For them to reject low quality nuts we need

$$V_P + x < V_T.$$

That is $V_T - V_P > x$. This implies

$$\frac{u}{x} > \frac{r+\alpha\phi+\gamma}{\gamma}$$

(h) Compare the parameter restrictions you obtained in parts d. and g. above. Is there a range of parameter values for which both equilibria can exist?

Multiplicity can occur whenever

$$\frac{r+\alpha+\phi\gamma}{\phi\gamma} > \frac{r+\alpha\phi+\gamma}{\gamma}$$

Cross multiplying and collecting terms implies that is true whenever

$$r(1-\phi)+\alpha(1-\phi^2)>0$$

which is always true. So both of these equilibria will coexist whenever

$$\frac{u}{x}\in\left(1+\frac{r+\alpha\phi}{\gamma},1+\frac{r+\alpha}{\phi\gamma}\right).$$

Within this parameter range, whenever everyone else is accepting both types of good so will I but if everyone else rejects low quality goods then it is in my interest to reject them too. It should be possible to Pareto rank these equilibria. Moreover, other types of equilibria are also possible. there is likely to be a mixed strategy equilibrium in which people randomize as to whether they accept the low quality good

2. **Sidrauski model with Proportional (Tooth Fairy) Money Distribution**

   Time: Discrete, infinite horizon, $t=0,1,2,...$

   Demography: A single representative infinite lived consumer/worker household. There is single representative firm owned by the household.

   Preferences: The instantaneous household utility function over, consumption, c_t, and real money balances, m_t, is $u(c_t,m_t)$. The function $u(.,.)$ is twice differentiable, strictly increasing in both arguments and strictly concave with usual Inada type conditions. The discount factor is $\beta\in(0,1)$.

   Technology: Aggregate output, $Y_t=F(K_t,L_t)$ where K_t is the aggregate capital stock and $L_t=1$ is the aggregate labor supply. The function $F(.,.)$ is twice differentiable, strictly increasing in both arguments, concave, exhibits constant returns to scale and the Inada conditions apply. It will be convenient to use $f(k_t)$ as the output per worker where k_t is the capital stock per worker. Capital depreciates by a factor δ in use each period.

   Endowments: Each household has one unit of labor and an initial endowment of capital k_0. Each also has an initial nominal money holding H_0.

   Information: Complete, perfect foresight.

   Institutions: Competitive markets in each period for capital, the consumption good, labor and money. There is a government that issues new money each period to maintain a fixed growth rate of the money supply: $H_t=(1+\sigma)H_{t-1}$. The money is distributed by transfer between periods to households in proportion, τ_t, to the amount they hold at the end of the previous period. Thus $M_t=M^d_{t-1}(1+\tau_t)$. The household takes τ_t as given.

(a) Using M_t^d as period t nominal money demand, P_t as the period t price of the consumption good, r_t as the rental rate on capital and w_t as the wage paid per unit of labor, write down and solve the household's problem (include the transversality condition).

$$V(k_t, M_t) = \max_{c_t, k_t, M_t^d} \left\{ u\left(c_t, \frac{M_t}{P_t}\right) + \beta V(k_{t+1}, M_{t+1}) \right\}$$

subject to

$$c_t = \frac{M_t}{P_t} - \frac{M_t^d}{P_t} + r_t k_t + w_t + (1-\delta)k_t - k_{t+1}$$

where $M_{t+1} = M_t^d(1+\tau_{t+1})$. After substituting out c_t the first order conditions are:

$$\begin{aligned}
k_{t+1} \quad &: \quad -u_1(c_t, m_t) + \beta V_1(k_{t+1}, M_{t+1}) = 0 \\
M_t^d \quad &: \quad -\frac{u_1(c_t, m_t)}{P_t} + \beta V_2(k_{t+1}, M_{t+1})(1+\tau_{t+1})
\end{aligned}$$

where $m_t = \frac{M_t}{P_t}$ is (post-transfer) real money balances brought into period t. The envelope conditions are:

$$\begin{aligned}
V_1(k_t, M_t) &= u_1(c_t, m_t)(r_t + 1 - \delta) \\
V_2(k_t, M_t) &= \frac{u_1(c_t, m_t)}{P_t} + \frac{u_2(c_t, m_t)}{P_t}.
\end{aligned}$$

Substituting into the FOC's yields

$$\begin{aligned}
-u_1(c_t, m_t) + \beta u_1(c_{t+1}, m_{t+1})(r_{t+1} + 1 - \delta) &= 0 \qquad (1) \\
-\frac{u_1(c_t, m_t)}{P_t} + \beta \left[\frac{u_1(c_{t+1}, m_{t+1})}{P_{t+1}} + \frac{u_2(c_{t+1}, m_{t+1})}{P_{t+1}}\right](1+\tau_{t+1}) &= 0 \qquad (2)
\end{aligned}$$

The transversality condition is

$$\lim_{t\to\infty} \beta^t u_1(c_t, m_t) k_t = 0$$

(b) Solve the problem faced by the firms, write down the market clearing conditions and the government budget constraint.
From the firm's problem, an interior solution requires that

$$\begin{aligned}
r_t &= f'(k_t^f) \\
w_t &= f(k_t^f) - k_t^f f'(k_t^f)
\end{aligned}$$

Market clearing requires:

$$\begin{aligned}
\text{Money} \quad &: \quad M_t^d = H_t = H_0(1+\sigma)^t \\
\text{Capital} \quad &: \quad k_t^f = k_t \\
\text{Goods} \quad &: \quad k_{t+1} + c_t = f(k_t) + (1-\delta)k_t
\end{aligned}$$

The government budget constraint is

$$H_{t+1} = (1+\tau_{t+1})H_t$$

(c) Define a monetary equilibrium and solve for the equations that characterize the equilibrium.

Definition: A Competitive monetary equilibrium is an allocation, $\left\{k_t, c_t, m_t, k_t^f\right\}$, prices, $\{P_t, r_t, w_t\}$, and transfers, $\{\tau_t\}$, such that (i) given prices and transfers the allocation solves the household's and firm's problems, (ii) markets clear and (iii) the government budget constraint holds.

The characterization is

$$\begin{aligned}
\frac{u_1(c_t, m_t)}{u_1(c_{t+1}, m_{t+1})} &= \beta(f'(k_{t+1}) + 1 - \delta) \\
u_1(c_t, m_t) &= \beta \frac{P_t}{P_{t+1}} \left[u_1(c_{t+1}, m_{t+1}) + u_2(c_{t+1}, m_{t+1})\right](1+\sigma) \\
k_{t+1} + c_t &= f(k_t) + (1-\delta)k_t
\end{aligned}$$

(d) Now obtain a set of equations that characterize the steady state, (k^*, c^*, m^*). Is money superneutral with respect to the "physical" steady-state allocation, (k^*, c^*)? Is it superneutral with respect to the entire steady-state allocation? Explain.

Note that in steady state $m_{t+1} = m_t$ for all t implies that $\frac{P_{t+1}}{P_t} = 1 + \sigma$. Then, steady-state equilibrium is characterized by

$$\begin{aligned}
f'(k^*) &= \frac{1-\delta}{\beta} \\
c^* &= f(k^*) - \delta k^* \\
u_1(c^*, m^*) &= \beta\left[u_1(c^*, m^*) + u_2(c^*, m^*)\right].
\end{aligned}$$

Both the physical allocation and the entire steady-state allocation are invariant to σ. Money is completely superneutral. Because money is distibuted proportionally to current holdings its effect is fully internalized by the saver.