Final Exam: Answer both questions time allowed 2 hours

1. **One-sided search with wage sensitive separation rates**

   In the data, separation rates are not invariant to the wage paid. Here we explore the implications of this for the one-sided search model.

   Time: Discrete, infinite horizon.

   Demography: A single infinite lived worker.

   Preferences: The worker is risk neutral (i.e. $u(w) = w$) and discounts the future at the rate r.

   Endowments: Each period while unemployed, the worker gets b units of the consumption good as a benefit and with probability α, the worker gets a wage offer, $w \sim F(.)$ on $[0, \bar{w}]$ with $\bar{w} > b$. When employed at the wage w, the worker receives w units of the consumption good each period but also faces lay-off with probability $\lambda(w)$ each period.

   (a) Write down the flow value equations for the worker.

   $$\begin{aligned} rV_e(w) &= w + \lambda(w)\left[V_u - V_e(w)\right] \\ rV_u &= b + \alpha \boldsymbol{E}_w \max\{V_e(w) - V_u, 0\} \end{aligned}$$

   (b) Show that the fundamental equation of search holds in this environment?

   $$V_e(w) = \frac{w + \lambda(w)V_u}{r + \lambda(w)}$$

   Defining w^* by $V_e(w^*) = V_u$ and evaluating $V_e(w)$ at w^* implies

   $$V_e(w^*) = \frac{w^* + \lambda(w^*)V_u}{r + \lambda(w^*)} = \frac{w^* + \lambda(w^*)V_e(w^*)}{r + \lambda(w^*)}$$

   then $rV_e(w^*) = rV_u = w^*$.

   (c) Obtain the reservation wage equation for this worker

   The usual analysis means

   $$w^* = b + \frac{\alpha}{r + \lambda(w^*)} \int_{w^*}^{\bar{w}} (w - w^*)\, dF(w)$$

   (d) How does the reservation wage change with α? (Provide the algebra to support your answer and explain how the dependency of separation on the wage affects the result.)

$$
\begin{aligned}
\frac{dw^*}{d\alpha} &= \frac{-\left[-\frac{1}{r+\lambda(w^*)}\int_{w^*}^{\bar{w}}(w-w^*)\,dF(w)\right]}{1-\frac{\alpha}{r+\lambda(w^*)}\left[\int_{w^*}^{\bar{w}}-dF(w)\right]+\frac{\alpha\int_{w^*}^{\bar{w}}(w-w^*)dF(w)}{[r+\lambda(w^*)]^2}} \\
&= \frac{[r+\lambda(w^*)]^2\,(w^*-b)}{\alpha\,[r+\lambda(w^*)]\,[r+\lambda(w^*)+\alpha(1-F(w^*)]+\alpha(w^*-b)\lambda'(w^*)} > 0
\end{aligned}
$$

If $\lambda'(.) > 0$, $\frac{dw^*}{d\alpha}$ is lower than when λ is constant. Increasing separation rates with the wage means that the workers will be less ready to raise their reservation wage as the job offer arrival rate rises. That's because the jobs are not that much better. They offer higher wages but also cause job loss more quickly.

2. **Optimal growth with a storage technology**

   Time: Discrete; infinite horizon

   Demography: A single representative (price taking) consumer/producer household.

   Preferences: the instantaneous household utility function over, individual consumption, c, is $u(c)$ where $u(.)$ is twice differentiable, strictly increasing and strictly concave. The discount factor is $\beta \in (0, 1)$.

   Productive Technology: There is a constant returns to scale technology over capital and labor such that output per unit of labor employed is $f(k)$, where k is capital input per unit of labor; $f(.)$ is twice differentiable, strictly increasing and concave with $\lim_{k\to 0} f'(k) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$. Capital depreciates at the rate $\delta < 1$.

   Storage Technology: The household can alternatively put goods into storage. If they store s_{t+1} units of the consumption good this period, they get back γs_{t+1} ($\gamma > 0$) units in the following period.

   Endowments: The households' initial capital stock is k_0, it also has 1 unit of labor each period. (We will focus on the Planner's problem so it doesn't matter whether they have access to the technology through firms or they have direct access.)

   (a) Write down the Planner's problem in recursive form (i.e. using value functions). **Note:** Because of the Inada conditions on $f(.)$ we do not have to worry about non-negativity of k but non-negativity of s can bind.

   The planner solves:

$$
\begin{aligned}
v(k_t, s_t) &= \max_{k_{t+1}, s_{t+1}, c_t} \{u(c_t) + \beta v(k_{t+1}, s_{t+1})\} \\
\text{s. t.} \quad c_t &= f(k_t) + (1-\delta)k_t - k_{t+1} + \gamma s_t - s_{t+1} \\
\text{and} \quad s_{t+1} &\geq 0
\end{aligned}
$$

   (b) Obtain a system of equations that can, in principle, be used to obtain the time paths of c_t, k_t and s_t.

   After substituting out c_t using the budget constraint and using μ_t as the multiplier on the non-negativity of storage constraint we get the following optimality conditions:

$$
\begin{aligned}
k_{t+1} &: \quad -u'(c_t) + \beta v_1(k_{t+1}, s_{t+1}) = 0 \\
s_{t+1} &: \quad -u'(c_t) + \beta v_2(k_{t+1}, s_{t+1}) + \mu_t = 0 \\
\mu_t s_{t+1} &= 0, \qquad \mu_t \geq 0
\end{aligned}
$$

   The envelope conditions are

$$
\begin{aligned}
v_1(k_t, s_t) &= u'(c_t)(f'(k_t) + 1 - \delta) \\
v_2(k_t, s_t) &= u'(c_t)\gamma
\end{aligned}
$$

After rolling forward and substituting for the value functions we get

$$\begin{aligned} -u'(c_t) + \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta) &= 0 \\ -u'(c_t) + \beta u'(c_{t+1})\gamma + \mu_t &= 0 \end{aligned}$$

The transversality condition will be

$$\lim_{t\to\infty} \beta^t u'(c_t)(k_t + s_t) = 0$$

(c) Temporarily assume that γ is very small so that we can ignore the storage technology (i.e. the economy reverts to that covered in class). Write down a set of equations that gives the non-trivial steady state of the system. What is the gross rate of return on capital in this steady state?

With $\gamma \approx 0$, the steady state, (c^*, k^*, s^*) will be given by:

$$\begin{aligned} f'(k^*) + 1 - \delta &= 1/\beta \\ c^* &= f(k^*) - \delta k^* \\ s^* &= 0 \\ \mu^* &> 0 \end{aligned}$$

The gross rate of return on every unit of capital created is therefore $1/\beta$.

(d) Now suppose that γ is strictly larger than the rate of return on capital you obtained in part (c). What happens to the steady state? Explain your answer.

If the rate of return on storage exceeds $1/\beta$ then the solution will include the use of the storage technology, so that $\mu_t = 0$. In that case the rate of return on capital will also equal γ this will mean that

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta\gamma > 1.$$

There is no steady state, consumption will grow forever. Such growth can still be consistent with the transversality condition. Because there is no curvature in the production function, the economy can grow forever. This is basically called the "AK" technology and is at the heart of all endogenous growth models.