AECO601 Fall 2015

Macroeconomics I

Final Exam

Answer both questions time allowed 2 hours

(1) Diamond cocoanut model with driftwood technology

Time: Discrete, infinite horizon.

Demography: A continuum, mass 1 of individuals

Geography: Two islands, trading and production

Preferences: Consumption of own output yields 0 utils, consumption of someone else's output yields u utils. There is a common discount rate r. Output is non-divisible and non-storable except by the original producer. There is a maximum inventory of 1.

Endowments: Everyone starts out with 1 unit of their own output – we will be focussed only on steady state allocations.

Productive Technology: To obtain cocoanuts everyone needs a stick (or pole) to knock the nuts off the trees on the production island. Some poles are longer than others. The length is characterized by the probability, α, that the pole can be used to remove nuts from any given tree. People seeking a nut find one tree every period, if they they have a pole of type α, it means that the probability they can obtain a nut from that tree is α. Obtaining a nut, given your pole is long enough, is costless.

Poles are found on the beach (as driftwood). When people arrive at the production island they search the beach for a pole. In any period, with probability σ they discover a pole. The lengths of the poles, characterized by α, are distributed $G(.)$ on $[0, 1]$. That means if you find a pole with $\alpha = 1$, you will get a cocoanut from every tree you encounter. At the other extreme, a (very short) pole with $\alpha = 0$ will never get you a nut. So, people search the beach for a pole they are happy with and then go to find trees with nuts they can reach. Once they produce (i.e. obtain a nut) they travel to the trading island. People cannot bring their pole with them and have to find a new one every time they want to produce.

Matching technology: On the trading island an individual meets another with probability γ each period.

Notes: Use the following notation: $V_P(\alpha)$, V_T, V_B to represent the values to being respectively on the production island looking for nuts with a pole type α, on the trading island holding one's own output, and on the production island beach looking for poles.

(a) Write down the flow asset value or Bellman equations associated with each state.
Flow asset equations are:

$$rV_T = \gamma(u + V_B - V_T)$$

$$rV_B = \sigma \mathbf{E}_\alpha \max\{V_P(\alpha) - V_B, 0\}$$

$$rV_P(\alpha) = \alpha(V_T - V_P(\alpha))$$

(b) Obtain an equation that implicitly specifies α^*, the reservation pole length, in terms of model parameters and $G(.)$.

The reservation pole length, α^*, solves $V_P(\alpha^*) = V_B$ so that

$$rV_B = \sigma \int_{\alpha^*}^{1} V_P(\alpha) - V_P(\alpha^*) dG(\alpha) \qquad (*)$$

Now,

$$V_P(\alpha) = \frac{\alpha V_T}{r+\gamma}$$

and

$$V_T = \frac{\gamma(u+V_B)}{r+\alpha}$$

so

$$V_P(\alpha) = \frac{\alpha\gamma(u+V_B)}{(r+\gamma)(r+\alpha)}.$$

Evaluating at α^* yields,

$$V_P(\alpha^*) = \frac{\alpha^*\gamma(u+V_P(\alpha^*))}{(r+\gamma)(r+\alpha^*)}.$$

Solving for $V_P(\alpha^*)$ we obtain

$$V_P(\alpha^*) = \frac{\alpha^*\gamma u}{r(r+\alpha^*+\gamma)}.$$

So

$$V_P(\alpha) - V_P(\alpha^*) = \frac{[\alpha\gamma(r+\alpha^*+\gamma) - \alpha^*\gamma(r+\alpha+\gamma)]\,u}{(r+\gamma)(r+\alpha)(r+\alpha^*+\gamma)}$$

which simplifies to

$$V_P(\alpha) - V_P(\alpha^*) = \frac{[\alpha-\alpha^*]\,\gamma u}{(r+\alpha)(r+\alpha^*+\gamma)}.$$

Into equation (*) yields

$$\alpha^* = \sigma \int_{\alpha^*}^{1} \left(\frac{\alpha-\alpha^*}{r+\alpha}\right) dG(\alpha)$$

(c) Obtain $\frac{d\alpha^*}{d\sigma}$, the effect of increasing the pole finding rate on the reservation pole length. Briefly explain your answer.

$$\frac{d\alpha^*}{d\sigma} = \frac{\int_{\alpha^*}^{1}\left(\frac{\alpha-\alpha^*}{r+\alpha}\right)dG(\alpha)}{1+\sigma\int_{\alpha^*}^{1}\left(\frac{1}{r+\alpha}\right)dG(\alpha)} = \frac{(\alpha^*/\sigma)}{1+\sigma\int_{\alpha^*}^{1}\left(\frac{1}{r+\alpha}\right)dG(\alpha)} > 0$$

Being able to find poles more frequently means that you can be more selective as to their length.
(d)What is $\frac{d\alpha^*}{d\gamma}$, the effect of increasing the trading island matching rate on the reservation pole length. Briefly explain your answer.

The answer is 0. There is no impact from changing the matching rate on α^*. On the one hand, once you have a nut an increased matching rate means you trade more quickly. This will make you want to search longer for a better pole to get to the trading island quicker. But, trading quicker means you will get another chance to find a good pole sooner this makes you less picky now about what pole to get. These effects cancel out.

(2) Optimal growth dynamics from technology changes

Consider the following Economy:

Time: Discrete; infinite horizon

Demography: Continuum of mass 1 of (representative) consumer/worker households, and a large number of profit maximizing firms. (We will focus only on the Planner's model.)

Preferences: the instantaneous household utility function over, individual consumption, c, is $u(c)$. Where $u(.)$ is twice differentiable, strictly increasing, strictly concave and $u'(0) = \infty$. The discount factor is $\beta \in (0, 1)$.

Technology: There is a constant returns to scale technology over capital and labor such that output per unit of labor employed is $zf(k)$, where k is capital input per unit of labor, z is the Total Factor Productivity (TFP) and $f(.)$ is twice differentiable, increasing and strictly concave with $f(0) = 0$, $f'(0) = \infty$ and $\lim_{k\to\infty} f'(k) = 0$. Capital depreciates at the rate $\delta < 1$.

Endowments: Each households has initial capital stock k_0 and 1 unit of labor.

(a) Write down and solve the Planner's problem for this economy (you can use either dynamic programming or calculus)

Problem is:

$$\max_{\{k_{t+1},c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t = zf(k_t) + (1-\delta)k_t - k_{t+1} \tag{\&}$$

Using calculus directly and after substituting for c_t the first order condition for k_{t+1} implies

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{\beta\left[zf'(k_{t+1}) + 1 - \delta\right]} \tag{\&\&}$$

(b) Obtain expressions for the steady state levels of consumption, c^*, and capital, k^*.

$$zf'(k^*) = \delta + \rho$$

where $\rho = \beta/(1-\beta)$ and

$$c^* = zf(k^*) - \delta k^*$$

(c) We will draw the phase diagram for this model. Show that

$$\begin{aligned} k_{t+1} &\geq k_t \Longleftrightarrow c_t \leq zf(k_t) - \delta k_t \\ c_{t+1} &\geq c_t \Longleftrightarrow c_t \geq zf(k_t) - \delta k_t + k_t - k^* \end{aligned}$$

From (&)

$$\begin{aligned} k_{t+1} &\geq k_t \Leftrightarrow zf(k_t) + (1-\delta)k_t - c_t \geq k_t \\ &\Leftrightarrow zf(k_t) + \delta k_t \geq c_t \end{aligned}$$

From (&&)

$$\begin{aligned} c_{t+1} &\geq c_t \Longleftrightarrow \beta u(c_{t+1}) \leq \beta u(c_t) \\ &\Longleftrightarrow \frac{u'(c_t)}{\beta\left[zf'(k_{t+1}) + 1 - \delta\right]} \leq \beta u(c_t) \\ &\Longleftrightarrow 1 \leq \beta\left[zf'(k_{t+1}) + 1 - \delta\right] \\ &\Longleftrightarrow k_{t+1} \leq k^* \\ &\Longleftrightarrow zf(k_t) + (1-\delta)k_t - c_t \leq k^* \\ &\Longleftrightarrow c_t \geq zf(k_t) - \delta k_t + k_t - k^* \end{aligned}$$

(d) Sketch the loci of points for which $k_{t+1} = k_t$ and $c_{t+1} = c_t$ in (k_t, c_t) space and draw on a saddle path.

(e) Now consider what happens if there is a one-time unforeseen negative and permanent technology shock. That is, prior to period $\hat{t}$, $z = z_0$ then after $t = \hat{t}$, $z = z_1 < z_0$.

(i) To see how the steady state is affected, obtain $\frac{dk^*}{dz}$ and $\frac{dc^*}{dz}$.

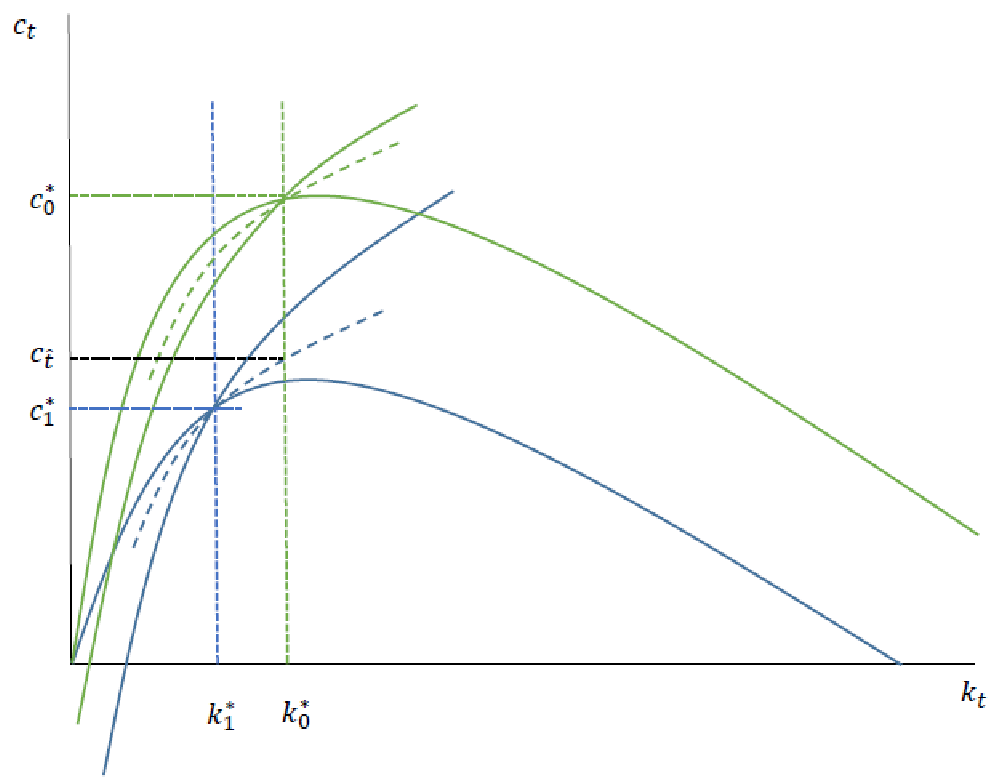
$$
\begin{aligned}
\frac{dk^*}{dz} &= \frac{-f'(k^*)}{zf''(k^*)} > 0 \\
\frac{dc^*}{dz} &= f(k^*) + \left[zf'(k^*) - \delta\right]\frac{dk^*}{dz} = f(k^*) - \rho\frac{f'(k^*)}{zf''(k^*)} > 0
\end{aligned}
$$

So both steady state values decrease when z decreases.

(ii) How are the loci of points for which $k_{t+1} = k_t$ and $c_{t+1} = c_t$ affected?

They both shift down when z decreases

(iii) Draw a phase diagram to show how the model predicts the paths of c_t and k_t starting from time $\hat{t}$.



The dashed curves are the saddle paths. Initially the economy is at (k_0^*, c_0^*). At time $\hat{t}$ it jumps to $(k_0^*, c_{\hat{t}})$. After this it tracks along the blue saddle path over time to end up at (k_1^*, c_1^*).