AECO601 Fall 2009

Macroeconomics I

Final Exam

Answer both questions time allowed 2 hours

(1) **Optimal Growth with Production Externalities.**

Time: Discrete; infinite horizon

Demography: Continuum of mass 1 of (representative) infinite lived consumer/worker households, and a large number of profit maximizing firms, owned jointly by the households.

Preferences: the instantaneous household utility function over, individual consumption, c, is $u(c)$ where $u(.)$ is strictly increasing and strictly concave. The discount factor is $\beta \in (0, 1)$.

Technology: Each firm has access to a technology which has constant returns to scale with respect to the capital and labor it hires. Total factor productivity however is increasing in the average level of capital in the economy. Thus output per unit of labor employed is $z(\bar{k})f(k^f)$ where $\bar{k}$ is average per capita capital stock, k^f is capital hired per unit of labor, $z(.)$ is total factor productivity (with $z(0) = 0$, $z'(.) > 0$) and f is strictly increasing and strictly concave with $f(0) = 0$. Aggregate production is therefore given by $\phi(\bar{k}) \equiv z(\bar{k})f(\bar{k})$. Assume that $\phi(.)$ is also strictly concave. Capital depreciates at the rate $\delta < 1$.

Endowments: Households' initial capital stock is k_0, each household has 1 unit of labor.

Institutions: Every period there are markets for capital and labor. (Firms are small and behave competitively).

(a) Write down and characterize the solution to the Planner's problem for this economy

The planner solves

$$\max_{\{k_{t+1}c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \text{ s.t. } \phi(k_t) + (1-\delta)k_t = c_t + k_{t+1}$$

or, in recursive form

$$V^P(k_t) = \max_{k_{t+1}} \left\{ u(\phi(k_t) + (1-\delta)k_t - k_{t+1}) + \beta V^P(k_{t+1}) \right\}$$

F.O.C.:

$$-u'(c_t) + \beta V^{P\prime}(k_{t+1}) = 0$$

Envelope:

$$V^{P\prime}(k_t) = u'(c_t)(\phi'(k_t) + 1 - \delta)$$

Leading to an Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})(\phi'(k_{t+1}) + 1 - \delta)$$

The transversality condition is

$$\lim_{t\to\infty} \beta^t u'(c_t)k_t = 0$$

(b) Write down and solve the household's problem given prices. (Hint you might want to express prices as functions of the aggregate capital stock, $\bar{k}_t$.)

Households solve

$$\begin{aligned} V(k_t, \bar{k}_t) &= \max_{c_t, k_{t+1}} \left\{u(c_t) + \beta V(k_{t+1}, \bar{k}_{t+1})\right\} \quad (1) \\ \text{subject to, } r(\bar{k}_t)k_t + (1-\delta)k_t + w(\bar{k}_t) &= c_t + k_{t+1} \\ \text{and } \bar{k}_{t+1} &= \arg\max_{k_{t+1}} \left\{u(c_t) + \beta V(k_{t+1}, \bar{k}_{t+1})\right\} \end{aligned}$$

or we could simply have

$$\begin{aligned} V(k_t) &= \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\} \\ \text{subject to, } r_t k_t + (1-\delta)k_t + w_t &= c_t + k_{t+1} \text{ given } r_t, w_t. \end{aligned}$$

(c) Write down and solve the firm's problem given prices.

Sticking with the formulation in (1), firms solve

$$\max_{k_t^f} \left\{z(\bar{k}_t)f(k_t^f) - r(\bar{k}_t)k_t^f - w(\bar{k}_t)\right\}$$

Which, along with constant returns to scale implies

$$\begin{aligned} r(\bar{k}_t) &= z(\bar{k}_t)f'(k_t^f) \\ w(\bar{k}_t) &= z(\bar{k}_t)f(k_t^f) - z(\bar{k}_t)f'(k_t^f)k_t^f \end{aligned}$$

Market clearing requires that $k_t = k_t^f$.

(d) Define a competitive equilibrium and solve for a general characterization.

Definition: A *competitive equilibrium* is an allocation, $\{k_{t+1}, c_t, k_t^f\}$, a pair of price function $r(.)$ and $w(.)$ and a sequence of aggregate capital stocks $\{\bar{k}_t\}$ such that, given the sequence of aggregate capital and the price functions, the allocation solves the household and firm's problems, markets clear and $\bar{k}_t = k_t$ for all t.

An alternative would be

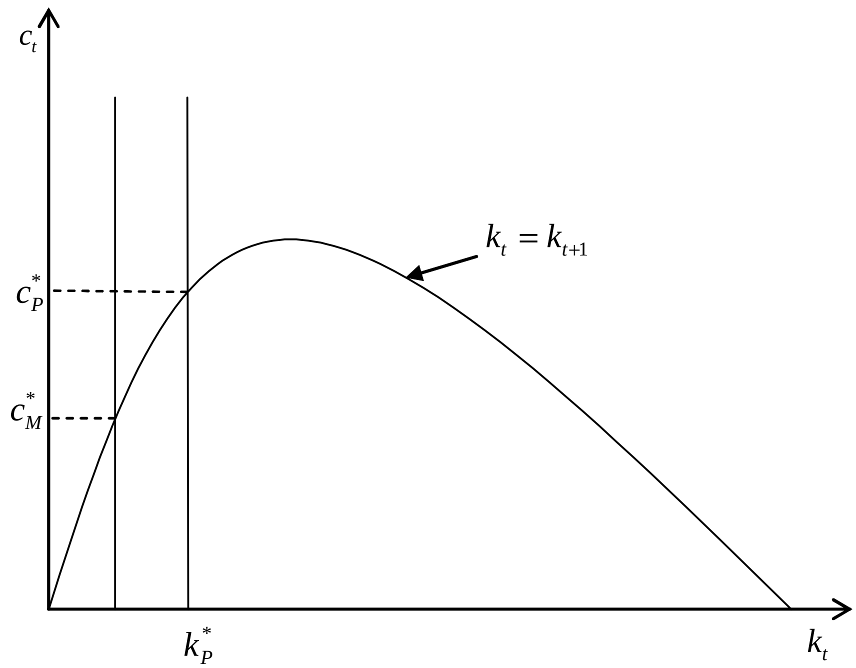

Figure 1: Phase diagram

Definition: A *competitive equilibrium* is an allocation, $\{k_{t+1}, c_t, k_t^f\}$ and a sequence of prices, $\{r_t, w_t\}$ such that given prices, the allocation solves the household and firm's problems and markets clear

Equilibrium is characterized by

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1})(z(k_t)f'(k_t) + 1 - \delta) \\ z(k_t)f(k_t) &= c_t + k_{t+1} - (1-\delta)k_t \end{aligned}$$

(e) Sketch the phase diagram for the planners's problem and for the market economy on the same axes. How do the (saddlepath stable) steady-state outcomes differ between the Planner's problem and the market economy? Explain.

In both systems the locus of points for which $k_t = k_{t+1}$ is

$$c = z(k)f(k) - \delta k$$

For the planner, $c_t = c_{t+1}$ implies

$$\beta \left[z(k)f'(k) + z'(k)f(k) + 1 - \delta \right] = 1$$

For the market economy, $c_t = c_{t+1}$ implies

$$\beta \left[z(k)f'(k) + 1 - \delta \right] = 1$$

So $k_P^* > k_M^*$ where k_P^* is the Planner's steady state capital stock and $c_P^* > c_M^*$.

(2) **Diamond Coconut Economy with Match-specific preferences**

Time: Discrete, infinite horizon

Geography: A trading island and a production island.

Demography: A mass of 1 of ex ante identical individuals with infinite lives

Preferences: The common discount rate is r, consumption of own produce yields 0 utils, consumption of anyone else's output yields $u \sim F$ on $[0, \bar{u}]$ utils.

Productive Technology: On the production island individuals come across a tree with a coconut with probability α each period. The cost of obtaining the coconut is 0, in every case.

Matching Technology: On the trading island people with coconuts meet another with probability γ, a constant.

Navigation: Travel between islands is instantaneous.

Endowments: Everyone has a boat and starts off with one of their own coconuts

(a) Assume to start with that everyone else accepts any coconut with probability Ω. Write down the asset value (Bellman's) equations for the problem faced by an individual in this environment. (Note: The individual believes that anyone he meets will take his coconut with probability Ω, however, the individual chooses whether or not to accept the coconut of anyone he meets based on the utility he will get from eating it.)

$$\begin{aligned} rV_P &= \alpha(V_T - V_P) \\ rV_T &= \gamma\Omega E_u[\max\{u + V_P - V_T, 0\}] \end{aligned} \tag{2}$$

(b) Let $\hat{u}$ represent the critical utility from consumption for which the individual is willing to give up his coconut. Express $\hat{u}$ in terms of the value to being on the trading island and the value to being on the production island.

$$\hat{u} = V_T - V_P$$

(c) Derive a reservation utility equation for the individual taking Ω as given. (It should not contain any endogenous variables other than $\hat{u}$ and Ω.)

$$rV_T = \gamma\Omega \int_{\hat{u}}^{\bar{u}} (u - \hat{u})\, dF(u)$$

from (2), $rV_P = \alpha\hat{u}$ and as $V_T = \hat{u} + V_P$,

$$rV_T = (r + \alpha)\hat{u}$$

Eliminating V_T yields

$$(r + \alpha)\hat{u} = \gamma\Omega \int_{\hat{u}}^{\bar{u}} (u - \hat{u})\, dF(u)$$

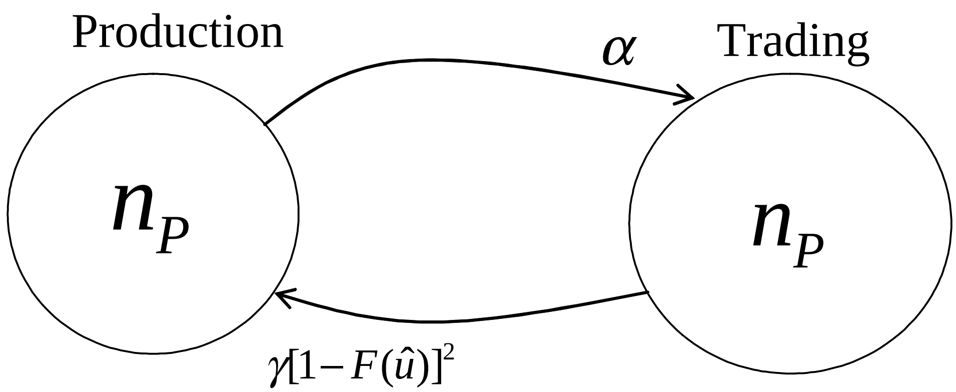

Figure 2: Steady state population flows

(d) Define steady-state symmetric equilibrium and provide an equation that characterizes the equilibrium reservation utility, u^*. (Hint express Ω in terms of the reservation utility of everyone else.)

Definition: A steady-state symmetric equilibrium is a reservation utility u^* such that when everyone else follows the implied trading strategy (i.e. accept when $u \geq u^*$ and reject otherwise) that u^* also represents each individual's optimal acceptance strategy.

In a symmetric steady-state equilibrium, $\Omega = 1 - F(u^*)$ so u^* is given by

$$(r+\alpha)u^* = \gamma\left[1 - F(u^*)\right] \int_{u^*}^{\bar{u}} (u - u^*)\, dF(u)$$

(e) Draw a diagram showing the population flows between the islands. Write down the steady state equations and solve for the population on the trading island as a function of u^*.

$$\begin{aligned} n_P + n_T &= 1 \\ \alpha n_P &= \gamma\left[1 - F(u^*)\right]^2 n_T \end{aligned}$$

So,

$$n_T = \frac{\alpha}{\alpha + \gamma\left[1 - F(u^*)\right]^2}.$$