AECO601 **Fall 2006**

FINAL EXAM

(1) Optimal Growth with a consumption externality.

Consider the following Economy:

Time: Discrete; infinite horizon

Demography: Continuum of mass 1 of (representative) consumer/worker households, and a large number of profit maximizing firms, owned jointly by the households.

Preferences: the instantaneous household utility function over, individual consumption, c, and average consumption $\bar{c}$ is $u(c) + v(\bar{c})$. Where $u(.)$ and $v(.)$ are both strictly increasing, $u(.)$ is strictly concave and $v(.)$ is concave. The discount factor is $\beta \in (0, 1)$. (Note: the sum of concave functions is concave.)

Technology: There is a constant returns to scale technology over capital and labor such that output per unit of labor employed is $f(k)$, where k is capital input per unit of labor. Capital depreciates at the rate $\delta < 1$.

Endowments: Households' initial capital stock is k_0, each household has 1 unit of labor.

(a) Write down the planner's problem in recursive form and solve for the system of equations that governs the dynamics of His/Her solution.

The planner does not distinguish between average and individual consumption, s/he solves

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}, c_t} \{u(c_t) + v(c_t) + \beta V(k_{t+1})\} \\ \text{s.t. } f(k_t) + (1-\delta)k_t &= c_t + k_{t+1} \end{aligned}$$

or

$$V(k_t) = \max_{k_{t+1}} \{u(f(k_t) + (1-\delta)k_t - k_{t+1}) + v(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta V(k_{t+1})\}$$

Which implies F.O.C:

$$u'(c_t) + v'(c_t) = \beta V'(k_{t+1})$$

Envelope condition:

$$V'(k_t) = [u'(c_t) + v'(c_t)]\,[f'(k_t) + (1-\delta)]$$

So system that determines the Planner's solution is:

$$\begin{aligned} \frac{u'(c_t) + v'(c_t)}{u'(c_{t+1}) + v'(c_{t+1})} &= \beta\,[f'(k_t) + (1-\delta)] \qquad (1) \\ f(k_t) + (1-\delta)k_t &= c_t + k_{t+1} \end{aligned}$$

(b) Now consider the decentralized economy. Write down the problem faced by the representative household, and the representative firm. Define and solve for a characterization of competitive equilibrium. (You can use either the sequence of markets or recursive competitive equilibrium.)

Individual household solves

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}, c_t} \{u(c_t) + v(\bar{c}_t) + \beta V(k_{t+1})\} \\ \text{s.t. } r_t k_t + w_t &= c_t + k_{t+1}, \quad \bar{c}_t = c_t \text{ (optimized value)} \end{aligned}$$

The latter constraint is a consistency requirement. It can be incorporated in many ways. What is important is the recognition that $\bar{c}_t$ is not directly controlled by the household.

The firm solves a profit maximization problem leading to:

$$r_t = \left[f'(k_t^f) + (1-\delta)\right], \quad w_t = f(k_t^f) - f'(k_t^f)k_t^f$$

where k_t^f is capital per worker hired by the firm.

Market clearing requires that $k_t^f = k_t$.

Definition: A *sequence of markets competitive equilibrium* is an allocation $\{c_t, k_t, k_t^f\}$ and a sequence of prices $\{r_t, w_t\}$ such that given prices c_t and k_t solve the households' problems, k_t^f solves the firms' problems and markets clear.

Characterization: solving the problems leads to

$$\begin{aligned} \frac{u'(c_t)}{u'(c_{t+1})} &= \beta\left[f'(k_t) + (1-\delta)\right] \\ f(k_t) + (1-\delta)k_t &= c_t + k_{t+1} \end{aligned} \tag{2}$$

(c) Compare the saddlepath-stable steady-states under the two different institutional arrangements in (a) and (b) (i.e. planner and competitive equilibrium) and comment on your answer.

The steady-states are the same. With additively separable utility across time all that matters for the steady state is the discount rate. While the consumption externality in general can distort consumption/savings decisions in steady state it does not. In a static model a positive consumption externality tends to lead to under consumption as individual households do not take account of the effect of the benefit of their own consumption on others' well being. Here, there is an additional effect, by saving a bit more individuals can effect the output available for aggregate consumption tomorrow. This implies that individuals may be consuming too much today. These spillovers just cancel out at the steady state.

(d) If $u(c) = \log(c)$ and $v(c) = c$, comment on the difference in the out-of-steady-state dynamics. In particular, which saddle path should be steeper? Comment on your answer.

We simply have to compare (1) to (2). System (1) becomes

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)\left(\frac{c_t+1}{c_{t+1}+1}\right) &= \beta\left[f'(k_t) + (1-\delta)\right] \\ f(k_t) + (1-\delta)k_t &= c_t + k_{t+1} \end{aligned} \tag{3}$$

and system (2) becomes

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right) &= \beta\left[f'(k_t) + (1-\delta)\right] \\ f(k_t) + (1-\delta)k_t &= c_t + k_{t+1} \end{aligned} \tag{4}$$

Consider a capital stock below k^*. In both systems $c_{t+1} > c_t$ but for a given level of k, $\left(\frac{c_{t+1}}{c_t}\right)$ in the planner's model has to be larger than in the decentralized economy so the planner's model saddlepath is steeper. For $k_t < k^*$ the savings effect dominates the consumption effect of the spillover. This is specific to the utility function. If $u(.)$ and $v(.)$ had the same functional form the Planner's model and the decentralized model would coincide at all values of k.

(2) **Diamond Coconut Economy with Fog**

Time: Discrete, infinite horizon

Geography: A single trading island and a large number of potential production islands. Each island has its own breed of tree that grow to an island specific height. So, all the coconut trees on any one island grow to the same height. The tree heights, c, across islands are distributed F. The support of F is $(0, \bar{c})$.

Demography: A mass of 1 of ex ante identical individuals with infinite lives

Preferences: The common discount rate is r, consumption of own produce yields 0 utils, consumption of anyone else's output yields $u > 0$ utils.

Productive Technology: On any island individuals come across a tree with a coconut with probability α each period. The cost of harvesting the coconut on an island with tree height c, is c utils.

Matching Technology: On the trading island people with coconuts meet another with probability γ.

Navigation: Travel to the trading island from any production island is instantaneous. Because of fog, finding a production island is a tricky business. People set out in boats but only hit a random island with probability σ each period. After arriving at an island the individuals have to decide whether to look for coconuts there or keep looking for a better island (i.e. one with shorter trees.)

Endowments: Everyone has a boat and starts off with a coconut

(a) Write down a set of Bellman or "asset value" equations that are implied by the above environment. (Hint: you will need 3 equations one each for: looking for trading partner, looking for a suitable island, looking for a coconut on the chosen island)

Let V_T be the value to being on the trading island, V_S be the value to looking for a suitable island, $V_P(c)$ is the value to looking for a nut on the chosen island. Note V_P will depend on c as it is calculated after choosing the island.

Bellman type equations:

$$\begin{aligned}
V_T &= \frac{1}{1+r}\left[\gamma(u+V_S)+(1-\gamma)V_T\right] \\
V_S &= \frac{1}{1+r}\left[\sigma\mathbb{E}_c \max\left\{V_P(c), V_S\right\}+(1-\sigma)V_S\right] \\
V_P(c) &= \frac{1}{1+r}\left[\alpha(V_T-c)+(1-\alpha)V_P(c)\right]
\end{aligned}$$

A possible alternative is

$$\begin{aligned}
V_T &= \frac{1}{1+r}\left[\gamma(u+V_S)+(1-\gamma)V_T\right] \\
V_S &= \frac{1}{1+r}\left[\sigma\mathbb{E}_c \max\left\{V_P(c), V_S\right\}+(1-\sigma)V_S\right] \\
V_P(c) &= \frac{1}{1+r}\left[\alpha \max(V_T-c, V_P(c))+(1-\alpha)V_P(c)\right]
\end{aligned}$$

The reason why I excluded the max operator from the from the first version is that because all the trees on the chosen island have the same height, there is no real choice at that stage. So, for these equations to be consistent with a non-trivial equilibrium, it must be true that $V_P(c) < V_T - c$ on any island that is chosen with positive probability.

Essentially what this set up does is to separate out the production cost decision (which occurs when an island is chosen) from the timing of production. The implied asset value equations are:

$$\begin{aligned}
rV_T &= \gamma(u+V_S-V_T) \\
rV_S &= \sigma\mathbb{E}_c \max\left\{V_P(c)-V_S, 0\right\} \\
rV_P(c) &= \alpha(V_T-c-V_P(c))
\end{aligned}$$

(b) Let c^* represent the critical tree height such that no one stops at an island where the trees are taller than c^*. Express c^* in terms of the value to being on the trading island and the value to looking for a suitable island. How should the value of c^* change with α, the proportion of trees with coconuts on any island? (If you do not have time to do the algebra make a guess and provide intuition for that guess.)

Defining the critical island tree height as c^* where $V_P(c^*) = V_S$ we can solve for a reservation tree height equation:

$$(r+\gamma)\, c^* = \gamma u - \frac{\sigma(r+\alpha+\gamma)}{r+\alpha} \int_0^{c^*} (c^* - c) dF(c)$$

Notice that if we put $\alpha = 1$ into this equation we do not get back the normal Diamond reservation cost equation. This is because the structure means that people have to pick the island in one period and, even with $\alpha = 1$, they have to wait until the next period before getting to produce. In a continuous time interpretation, α is the arrival rate of production opportunities. Putting $\alpha = \infty$ does indeed yield the original equation.

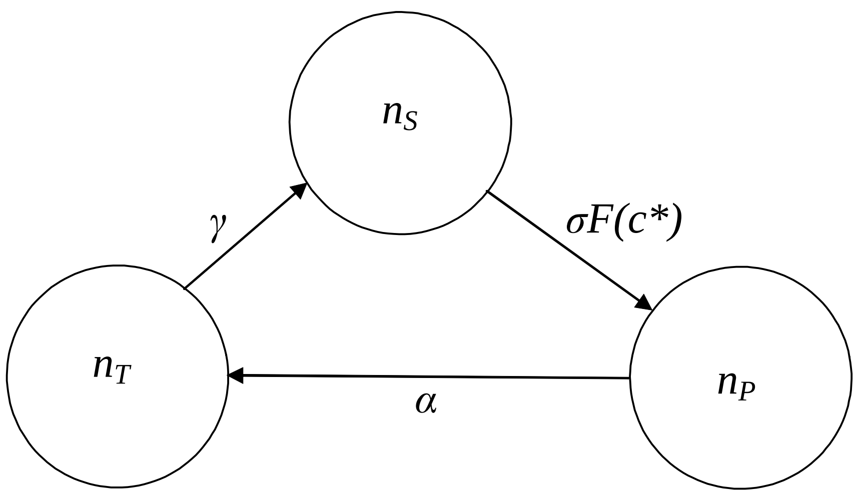
It is straightforward to show that increasing α increases c^*. This is because individuals expected wait time to produce falls as α increases. This allows people looking for islands to be pickier.

(c) Draw a diagram showing the flows between the states an individual can be in. Write down the steady state equations that could be used to figure out how many people are in each state.

Let n_i be the steady-state population in state i. Then:

Equations:

$$\begin{aligned} \gamma n_T &= \sigma F(c^*) n_S \\ \sigma F(c^*) n_S &= \alpha n_P \\ n_T + n_S + n_P &= 1 \end{aligned}$$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