TMAT/AMAT 119 Practice Midterm 2

Name: ____________________________

Information: Solutions to this practice exam will not be collected. The exam may take you longer than today's class time, and you are welcome to finish at home. Solutions will be posted online this weekend, and you are welcome to come discuss the exam at office hours, including those after this class.
For this practice exam, you can use your notes. For the actual exam, you may bring one page of handwritten notes (front and back). **Note that I will not be as generous with hints on the actual exam and you should come prepared with the necessary formulae.** No calculators, phones, etc. will be allowed. The actual exam may be longer than this one.

1. Evaluate $\int \sin^3 x \; dx$.

Answer: $\int \sin^3 x \; dx = \int \sin^2 x \;\; \sin x \; dx = \int (1-\cos^2 x) \sin x \; dx$.
$u = \cos x, \; -du = \sin x \; dx$.
$\int (1-\cos^2 x) \sin x \; dx = -\int (1-u^2) \; du = -u(x) + \frac{1}{3} u(x)^3 + C =$
$\frac{1}{3} \cos(x)^3 - \cos(x) + C$.

2. Evaluate $\int \sin^4 x \; dx$. [HINT: Use the half-angle formula $\sin^2 x = \frac{1}{2}(1-\cos 2x)$.]
Answer:

$$\begin{aligned}
\int \sin^4 x \; dx &= \frac{1}{4} \int (1-\cos 2x)^2 \; dx \\
&= \frac{1}{4} \int (1 - 2\cos 2x + \cos^2 2x) \; dx \\
&= \frac{1}{4} \int (2 - 2\cos 2x - \sin^2 2x) \; dx \\
&= \frac{1}{4} \int (2 - 2\cos 2x - \frac{1}{2} + \frac{1}{2} \cos 4x) \; dx \\
&= \frac{1}{4} \int (\frac{3}{2} - 2\cos 2x + \frac{1}{2} \cos 4x) \; dx \\
&= \frac{1}{4} \left(\frac{3x}{2} - \sin 2x + \frac{1}{8} \sin 4x + C \right) \\
&= \frac{3x}{8} - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C)
\end{aligned}$$

3. Evaluate $\int \sin 3x \cos 2x \ dx$. [HINT: $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$.

Answer: $A = 3x,\ B = 2x$

$$\int \sin 3x \cos 2x \ dx = \frac{1}{2}\int [\sin x + \sin 5x] \ dx = -\cos x - \frac{1}{5}\cos 5x + C.$$

4. Compute $\int \frac{x^3}{\sqrt{x^2+4}} dx$. [HINT: use substitution $x = 2\tan\theta,\ \theta \in (-\pi/2, \pi/2)$.]

Answer: $x = 2\tan\theta,\ dx = 2\sec^2\theta \ d\theta$

$$\begin{aligned}
\int \frac{x^3}{\sqrt{x^2+4}} dx &= \int \frac{(8\tan^3\theta)(2\sec^2\theta)}{\sqrt{4\tan^2\theta+4}} d\theta \\
&= 8\int \frac{\tan^3\theta\sec^2\theta}{\sqrt{\tan^2\theta+1}} d\theta \\
&= 8\int \frac{\tan^3\theta\sec^2\theta}{\sec\theta} d\theta \\
&= 8\int \tan^3\theta\sec\theta \, d\theta \\
&= 8\int \tan^2\theta(\tan\theta\sec\theta) \, d\theta \\
&= 8\int (\sec^2\theta - 1)(\tan\theta\sec\theta) \, d\theta
\end{aligned}$$

[Use substitution: $u = \sec\theta,\ du = \sec\theta\tan\theta \ d\theta$.]

$$\begin{aligned}
= 8\int (\sec^2\theta - 1)(\tan\theta\sec\theta) \, d\theta &= 8\int (u^2-1) \, du \\
&= 8\left(\frac{1}{3}u(\theta)^3 - u(\theta)\right) + C \\
&= 8\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C \\
&= 8\left(\frac{1}{3}\sec^3\theta - \sec\theta\right) + C
\end{aligned}$$

Now write this in terms of x. $\tan\theta = x/2$. Consider the right triangle with angle θ, opposite side length x, and adjacent side length 2. The hypotenuse has length $\sqrt{x^2+4}$. So $\sec\theta = hyp/adj = \frac{\sqrt{x^2+4}}{2}$. Thus

$$\int \frac{x^3}{\sqrt{x^2+4}} dx = 8\left(\frac{1}{3}\right)\left(\frac{1}{8}\right)(x^2+4)^{3/2} - \frac{8\sqrt{x^2+4}}{2} + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C.$$

5. Compute $\int \frac{x}{\sqrt{x^2+4}} dx$. [HINT: Trig substitution is not the best approach here!]

Answer: $u = x^2+4,\ \frac{1}{2}du = x,$.

$$\int \frac{x}{\sqrt{x^2+4}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,dx = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C = \frac{1}{3}(x^2+4)^{3/2} + C.$$

6. Explain what it means for the improper integral $\int_0^\infty f(x)\,dx$ to be convergent. Use a limit in your answer.

Answer: $\lim_{b\to\infty} \int_0^b f(x)\,dx$ exists, as a finite number.

7. Compute $\int_1^\infty \frac{1}{x^3}\,dx$. (do not just use a formula for this, but compute this directly.)

Answer: $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty} -\frac{1}{2b^2} + \frac{1}{2} = \frac{1}{2}.$

8. Compute $\int_{-\infty}^0 \frac{1}{1+x^2}\,dx$. [HINT: $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$.]

Answer:

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \left[\tan^{-1}(x)\right]_a^0 = 0 - \frac{-\pi}{2} = \frac{\pi}{2}.$$

9. Determine whether the improper integral $\int_{-1}^1 \frac{1}{x^{1/3}}\,dx$ is convergent or divergent. If it is convergent, evaluate it.

Answer: We have an infinite discontinuity at 0, so by definition,

$$\int_{-1}^1 \frac{1}{x^{1/3}}\,dx = \int_{-1}^0 \frac{1}{x^{1/3}}\,dx + \int_0^1 \frac{1}{x^{1/3}}\,dx,$$

where both of the integrals on the right hand side are improper.

$$\int_{-1}^0 \frac{1}{x^{1/3}}\,dx = \lim_{a\to0^-} \int_{-1}^a \frac{1}{x^{1/3}}\,dx = \frac{3}{2}\lim_{a\to0^-} \left[x^{2/3}\right]_{-1}^a = 0 - \frac{3}{2} = -\frac{3}{2}.$$

A very similar calculation shows that $\int_0^1 \frac{1}{x^{1/3}}\,dx = \frac{3}{2}$. Hence $\int_{-1}^1 \frac{1}{x^{1/3}}\,dx$ is convergent, and $\int_{-1}^1 \frac{1}{x^{1/3}}\,dx = -\frac{3}{2} + \frac{3}{2} = 0.$

(It was clear from the outset that if the integral was convergent, then its value was zero, because the integrand is odd, and the bounds of integration are symmetric. However, we in any case had to do this work to show that the integral is convergent.)

10. Find the length of the arc $x = y^{3/2}$ between the points $(1,1)$ and $(27,9)$.

Answer: $L = \int_1^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$

$\frac{dx}{dy} = \frac{3}{2}\sqrt{y}$, so

$L = \int_1^9 \sqrt{1 + \frac{9y}{4}}\,dy.$

$u = 1 + \frac{9}{4}y$, $\frac{4}{9}du = dy$

$$L = \frac{4}{9}\int_{13/4}^{82/4} \sqrt{u}\, du = \left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(\left(\frac{82}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right) =$$
$$\frac{8}{27}\left(\left(\frac{82}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right).$$

11. Consider the surface obtained by rotating the curve of $y = \frac{1}{3}x^3$ from $x = 0$ to $x = 1$ about the y-axis. Write down an integral representing its surface area (you don't need to compute it though).

Answer: $y' = x^2$, so Area $= 2\pi \displaystyle\int_0^1 x\sqrt{1+x^4}\, dx$.

12. Compute the center of mass of the region R in the right half-plane bounded by the curves $y = x^2$ and $y = x^3$. **Answer**: First, let's compute the area of the region. The curves intersect at 0 and 1. Between 0 and 1, the first curve lies above the second. Hence, $A = \displaystyle\int_0^1 x^2 - x^3\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$.
By the formula on page 564 of Stewart, the center of mass is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_0^1 x(x^2 - x^3)\, dx = 12\int_0^1 x^3 - x^4\, dx = 12\left(\frac{x^4}{4} - \frac{x^5}{5}\right) = \frac{12}{20} = \frac{3}{5},$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left\{(x^2)^2 - (x^3)^2\right\}\, dx = \frac{12}{2}\int_0^1 x^4 - x^6\, dx = 6\left(\frac{x^5}{5} - \frac{x^7}{7}\right) = (6)\left(\frac{2}{35}\right) = \frac{12}{35}.$$

Thus $(\bar{x}, \bar{y}) = (\frac{3}{5}, \frac{12}{35})$.

13. Let X be a random variable with probably density function f.

(i) What is $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx$?
(ii) Express the probability $\Pr(0 \le X \le 1)$ in terms of f.
(iii) Express the probability $\Pr(X \le 1)$ in terms of f.
(iv) Express the mean of X in terms of f.

Answer:
(i) 1.
(ii) $\displaystyle\int_0^1 f(x)\, dx$.
(iii) $\displaystyle\int_{-\infty}^1 f(x)\, dx$.
(iv) $\displaystyle\int_{-\infty}^{\infty} x f(x)\, dx$.