TMAT/AMAT 119 Practice Midterm 1

Name: ___________________________

Information: Solutions to this practice exam will not be collected. The exam may take you longer than today's class time, and you are welcome to finish at home. Solutions will be posted online this weekend, and you are welcome to come discuss the exam at office hours, including those after this class.
For this practice exam, you can use your notes. For the actual exam, you may bring one page of handwritten notes. No calculators, phones, etc. will be allowed. The actual exam may be longer than this one.

1. Compute the area bounded by $y = \sin(x)$ and $y = \cos(x)$ from $x = 0$ to $x = \pi/2$.
Answer: The curves intersect at $\pi/4$, so we need to break up the computation into two parts. For $x \in [0, \pi/4]$, we have $\sin(x) \leq \cos(x)$, and for $x \in [\pi/2, \pi/2]$, we have $\cos(x) \leq \sin(x)$. Thus, the area A is given by

$$A = \int_0^{\pi/4} [\cos(x) - \sin(x)]\,dx + \int_{\pi/4}^{\pi/2} [\sin(x) - \cos(x)]\,dx$$

$$[\sin(x) + \cos(x)]_0^{\pi/4} - [\sin(x) + \cos(x)]_{\pi/4}^{\pi/2} =$$

$$(\sqrt{2}/2 + \sqrt{2}/2) - (0 + 1) - (1 + 0) + (\sqrt{2}/2 + \sqrt{2}/2) = 2\sqrt{2} - 2.$$

2. Compute the area bounded by the curves $y = \sqrt{x-1}$ and $x - y = 1$.
Answer: First, we compute where the curves intersect, by solving $\sqrt{x-1} = x - 1$. To solve this, we can square both sides, to get

$$x - 1 = x^2 - 2x + 1 \qquad \Longrightarrow \qquad 0 = x^2 - 3x + 2.$$

Clearly the solutions to this are $x = 1, 2$. For $x \in [1, 2]$, $\sqrt{x-1} \geq x - 1$ (consider, e.g., the test point 1.5), so we have that the area A is given by:

$$A = \int_1^2 [\sqrt{x-1} - (x-1)]\,dx = [2/3(x-1)^{3/2} - x^2/2 + x]_1^2 =$$

$$(2/3 - 2 + 2 + 1/2 - 1) = 1/6.$$

3. Consider the solid S obtained by rotating the region bounded by $y^2 = x$ and $x = 2y$ about the y-axis. Compute the volume of S.
Answer: I will use the washer method, integrating with respect to y. the two given curves intersect at the solutions to $y^2 - 2y = 0$, i.e. at $y = 0, 2$. For $y \in [0, 2]$, we have $2y \geq y^2$.
So

$$Vol(S) = \int_0^2 \pi((2y)^2 - (y^2)^2\,dy$$

$$= \pi[4y^3/3 - y^5/5]_0^2 = 32/3 - 32/5 = 64\pi/15.$$

4. Consider the solid S obtained by rotating the region bounded by $y = x^2$, $y = 0$ and $x = 2$ about the line $x = 4$. Compute the volume of S using the method of cylindrical shells.

Answer: According to the formula I gave in class,

$$Vol(S) = \int_0^2 2\pi|x-4|x^2\,dx = 2\pi\int_0^2 (4-x)x^2\,dx$$

$$= 2\pi\int_0^2 [4x^2 - x^3]\,dx = 2\pi[4x^3/3 - x^4/4]_0^2 = 2\pi(32/3 - 16/4).$$

5. Consider the solid S obtained by rotating the region bounded by $x = \sin(y)$, $y = 0$ and $x = \pi/4$ about the line $x = \pi$. Write down an integral for the volume of this solid using the washer method. You do not have to compute this integral, just write it down (and explain your answer).

Answer: Since we are rotating around a vertical line, to use the washer method we have to integrate with respect to y. We integrate over the interval $[0, \sin^{-1}(\pi/4)]$. For $y \in [0, \sin^{-1}(\pi/4)]$, we have $\sin(y) \leq \pi/4 \leq \pi$. Hence, the outer radius of the washer at y is $\pi - \sin(y)$ and the inner radius is $\pi - \pi/4 = 3\pi/4$. Thus, the area of the washer at y is $\pi[(\pi - \sin y)^2 - (3\pi/4)^2]$.

$$Vol(S) = \int_0^{\sin^{-1}(\pi/4)} \pi[(\pi - \sin y)^2 - (3\pi/4)^2]\,dy.$$

6. Compute $\int_{-3}^{-2} x\sqrt{x+3}\,dx$. HINT: First find an expression for $\int (x+3)^n\,dx$, for $n \neq 1$.

Answer: (I use the hint implicitly in this answer.) Use integration by parts:
$u = x,\ dv = \sqrt{x+3}$.
$du = dx,\ v = \frac{2(x+3)^{3/2}}{3}$

$$\int_{-3}^{-2} x\sqrt{x+3}\,dx = [uv]_{-3}^{-2} - \int_{-3}^{-2} v\,du = [2x(x+3)^{3/2}/3]_{-3}^{-2} - \frac{2}{3}\int_{-3}^{-2}(x+3)^{3/2}\,dx$$

$$= \left[\frac{2x(x+3)^{3/2}}{3}\right]_{-3}^{-2} - \left[\frac{4(x+3)^{5/2}}{15}\right]_{-3}^{-2} = -4/3 - 0 - 4/15 + 0 = -24/15 = -8/5.$$

7. Find a value of k such that the average value of $f(x) = x^3 + k$ over the interval $[0, 2]$ is 10.

Answer:

$$10 = f_{avg} = \frac{1}{2-0}\left[\frac{x^4}{4} + kx\right]_0^2 = \frac{4+2k}{2},$$

so $k = 8$.

8. Compute $\int_1^7 \frac{\ln x}{\sqrt{x}}\,dx$.

Answer: Use integration by parts:
$u = \ln x,\ dv = 1/\sqrt{x}$.
$du = dx/x,\ v = 2\sqrt{x}$

$$\int_1^7 \frac{\ln x}{\sqrt{x}}\,dx = \int_{-3}^{-2} x\sqrt{x+3}\,dx = [uv]_1^7 - \int_1^7 v\,du = 2[\ln x\sqrt{x}]_1^7 - 2\int_1^7 \frac{1}{\sqrt{x}}\,dx$$

$2[\ln x\sqrt{x}]_1^7 - 4[\sqrt{x}]_1^7 = 2\ln 7\sqrt{7} - 4\sqrt{7} + 4.$

10. Assuming f and g are continuous, and h is differentiable, simplify the following expression: $\frac{d}{dx}\int_a^{h(x)} [f(g(t)) + c]\ dt$. [Hint: Use the chain rule and FTC part 1.]

Answer: Let $q(y) = \int_a^y [f(g(t)) + c]\ dt$. Then $\int_a^{h(x)} [f(g(t)) + c]\ dt = q(h(x))$. By the FTC, $q'(y) = f(g(y)) + c$. Using this and the chain rule, we find that:

$$\frac{d}{dx}\int_a^{h(x)} [f(g(t)) + c]\ dt = (f(g(h(x))) + c)h'(x).$$

11. Compute $\int xe^{-x^2}\ dx$.

Answer: Use substitution:
$u = -x^2,\ -1/2du = xdx$.
$\int xe^{-x^2}\ dx = -1/2\int e^u\ du = -e^{u(x)}/2 + C = -e^{-x^2}/2 + C.$

12. Compute $\int_0^1 t^2\sqrt{2t^3+1}\,dt$.

Answer: Use substitution:
$u = 2t^3 + 1,\ 1/6du = t^2dt$.
$\int_0^1 t^2\sqrt{2t^3+1}\,dt = 1/6\int \sqrt{u}\,du = (1/6)(2/3)u(t)^{3/2} + C = \frac{(2t^3+1)^{3/2}}{9} + C$