TMAT/AMAT 118 Partial Practice Final

Name: ____________________________

Note: As a reminder, the actual final is cumulative, and will also cover material from the first two midterms. This partial practice final only covers material from after the cutoff for the second midterm. That material will account for roughly 40-50% of the exam.

1. Let $\{a_n\}$ and $\{b_n\}$ be sequences. Suppose $\lim_{n\to\infty}\{a_n\} = -1$ and $\lim_{n\to\infty}\{b_n\} = 1$. What can be said about $\lim_{n\to\infty}\{a_n + b_n\}$?

Answer: 0. According to one of the rules of limits of sequences, the sum of the limits is equal to the limit of the sums.

2. Let $\{a_n\} = a_1, a_2, a_3, \ldots$ be a sequence with $\lim_{n\to\infty}\{a_n\} = 2$, and let $\{b_n\}$ be the sequence given by

$$b_n = \begin{cases} a_n & \text{for } n > 1, \\ 0 & \text{for } n = 1 \end{cases}.$$

What is $\lim_{n\to\infty} b_n$?

Answer: 1. Changing a finite number of terms in a sequence does not affect the limit of the sequence, because the limit depends only on the behavior of the sequence for very large n.

3. For each condition, say whether the series $\sum_{n=1}^{\infty} a_n$ converges, diverges, or you can't tell. Explain your answer.

- $\sum_{n=1}^{\infty} -a_n$ converges. **Answer**: Converges: $\sum_{n=1}^{\infty} -a_n = -\sum_{n=1}^{\infty} a_n$
- $\lim_{n\to\infty} a_n \neq 0$, **Answer**: Diverges by the divergence test.
- $\lim_{n\to\infty} a_n = 0$, **Answer**: Can't tell. For example, a p-series diverges when $p = 1$ and converges when $p = 2$.
- $\sum_{n=1}^{\infty} |a_n|$ converges, **Answer**: Converges: An absolutely convergent series always converges.

- $\sum_{n=1}^{\infty} |a_n|$ diverges. **Answer**: Can't tell: Could be divergent (e.g. harmonic series) or conditionally convergent (e.g. alternating harmonic series.)

4. Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ converge or diverge? Explain your answer.

Answer: Converges by the alternating series test: The series is alternating with $b_n = \frac{1}{\sqrt{n+1}}$. There are two conditions to check:
First, $b_{n+1} = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = b_n$.
Second, $\lim_{n\to\infty} b_n = \frac{1}{\sqrt{n+1}} = 0$.

5. Does $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converge or diverge? Explain your answer.

Answer: Converges by the comparison test.
Compare with the p-series $\sum_{n=1}^{\infty} = \frac{1}{n^3}$. We have that $0 \leq \frac{1}{n^3+1} \leq \frac{1}{n^3}$ for all n, and $\sum_{n=1}^{\infty} = \frac{1}{n^3}$ is convergent, so $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$ must be convergent.

6. Does $\sum_{n=1}^{\infty} \frac{1}{n^3-1}$ converge or diverge? Explain your answer.

Answer: Converges by the limit comparison test. Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^3-1}$. As in the last problem, compare with the convergent p-series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} = \frac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^3-1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3-1} = 1 > 0.$$

Hence by the limit comparison test, either both series diverge, or both converge. Since $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

7. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{4^n}$? Explain your answer.

Answer: $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{-x}{4}\right)^n$. This is a geometric series with $r = -\frac{x}{4}$. It converges when $|r| = |\frac{x}{4}| < 1$, i.e. when $|x| < 4$, and diverges otherwise. Hence the radius of convergence is 4 and the interval of convergence is $(-4, 4)$.

8. Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^4}$? Explain your answer.

Answer: Most of the problems we saw of this kind use the ratio test (occasionally, you use the root test.). We will use the ratio test here.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^4} \cdot \frac{n^4}{x^n}\right| = |x| \frac{n^4}{(n+1)^4}.$$

Thus,

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| \frac{n^4}{(n+1)^4} = |x| \lim_{n\to\infty} \frac{n^4}{(n+1)^4} = |x|$$

for all x. By the ratio test, the series converges when $|x| < 1$ and diverges when $|x| > 1$. Hence, the radius of convergence is 1. To determine the interval of convergence, we need to consider the cases $x = \pm 1$ separately. When $x = 1$, the series is a p-series with $p = 4 > 1$, so is convergent. When $x = -1$ the series is $\sum_{n=1}^{\infty} \frac{-1^n}{n^4}$? The series of absolute values of terms is the p-series $\sum_{n=1}^{\infty} \frac{1^n}{n^4}$, which we just saw was convergent. An abolsolutely convergent series is convergent, so the series converges at $x = -1$ as well.
Hence the interval of convergence is $[-1, 1]$.

9. Find an infinite series expression for $\frac{1}{2-x}$. ADDED: and find its radius of convergence.

Answer: Write

$$\frac{1}{2-x} = \left(\frac{1}{2}\right)\left(\frac{1}{1-x/2}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^n}.$$

By the result about convergence of a geometric series, the series converges when $|\frac{x}{2}| < 1$, i.e., when $|x| \leq 2$. Hence, the radius of convergence is 2.

10. Compute the first two non-zero terms of Taylor series for $\sin x$ centered around $a = \pi/2$. Do this computation directly, don't just quote a formula.

Answer:

$$f^{(0)}(\pi/2) = \sin(\pi/2) = 1.$$
$$f^{(1)}(\pi/2) = \cos(\pi/2) = 0.$$
$$f^{(2)}(\pi/2) = -\sin(\pi/2) = -1.$$

The Taylor series for $\sin x$ at $\pi/2$ is

$$f^{(0)} + \frac{f^{(1)}(x-\pi/2)}{1!} + \frac{f^{(2)}(x-\pi/2)^2}{2!} + \cdots = 1 + 0 - \frac{(x-\pi/2)^2}{2} + \cdots .$$

So the first two non-zero terms are 1 and $\frac{(x-\pi/2)^2}{2}$.

11. Is the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+2}}$ convergent or divergent? If convergent, find the value. If divergent, explain why.

Answer: We have

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+2}} = \frac{1}{8}\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}.$$

$\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$ is a geometric series with $a = 1$ and $r = \frac{1}{2}$. Since $|r| < 1$, this geometric series is convergent. So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n+2}}$ is convergent as well, and its value is $\frac{1}{8} \cdot \frac{1}{1-1/2} = \frac{1}{4}$.