TMAT/AMAT 118 Practice Midterm 2

Name: ____________________________

1. Given a function $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, what does it mean for f to be continuous at a?
Answer: It means that $\lim_{x \to a} f(x) = f(a)$.

2. Let

$$f(x) = \begin{cases} 1/x & \text{if } x < 3, \\ 3x & \text{if } x \geq 3. \end{cases}$$

At which points is f not continuous?
Answer: $x = 0$ and $x = 3$.

3. Suppose f is continuous at a, and

$$\lim_{x \to a^+} f(x) = c.$$

What is $f(a)$?
Answer: $f(a) = c$, because $f(a) = \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = c$.

4. Give the definition of the derivative of a function f at a point a, in two different ways.
Answer:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

5. Using the definition of the derivative and basic properties of limits, prove that if f and g are both differentiable at a, then

$$(f+g)'(a) = f'(a) + g'(a).$$

Answer:

$$\begin{aligned}
(f+g)'(a) &= \lim_{x\to a} \frac{(f(x)+g(x)) - (f(a)+g(a))}{x-a} \\
&= \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a}\right] \\
&= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= f'(a) + g'(a).
\end{aligned}$$

6. Compute the derivative of $f(x) = \sqrt{x}$ directly from the definition of the derivative. Do not use the power rule.

Answer:

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \\
&= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \\
&= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} \cdot \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} \\
&= \lim_{x\to a} \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} \\
&= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} \\
&= \frac{1}{2\sqrt{a}}.
\end{aligned}$$

7. Complete the following to give the product rule for derivatives: If f and g are both functions which are differentiable everywhere, then $(fg)' =$ _________.
Answer: $(fg)' = gf' + fg'$.

8. Find the first and second derivatives of $f(x) = 3x^3 + 2x^2 + x$.
Answer: $f'(x) = 9x^2 + 4x + 1$, $f''(x) = 18x + 4$.

9. For $f(x)$ as in the previous problem, find the slope of the tangent line to the curve $y = f(x)$ at $(1, 6)$.
Answer: The slope is $f'(1) = 14$.

10. What is the second derivative of $f(x) = \cos x$?
Answer: $f'(x) = -\sin x$, so $f''(x) = -\cos x$.

11. What is the tenth derivative of $f(x) = \sin x$?
Answer: To solve this, note that the second derivative of $\sin x$ is -$\sin x$. Therefore, the fourth derivative of $\sin x$ is $\sin x$. Thus, the $(4k+j)^{th}$ derivative of $\sin x$ is equal to the j^{th} derivative. Taking $k = j = 2$, we find that the 10th derivative of $\sin x$ is equal to the second derivative of $\sin x$. Thus the answer is $-\sin x$.

12. Complete the following to give the chain rule: If g is differentiable at a and f is differentiable at $g(a)$, then $(f \circ g)'(a) =$ _________.
Answer: $(f \circ g)'(a) = f'(g(a))g'(a)$.

13. What are the first and second derivatives of $f(x) = \tan x$?
Answer:

$$f'(x) = \sec^2(x) = \frac{1}{\cos^2 x} = \cos^{-2}(x).$$

Applying the chain rule and the power rule, we have that

$$f''(x) = (-2)(\cos^{-3} x)(-\sin x) = \frac{2\sin x}{\cos^3 x} = 2\tan x \sec^2 x.$$

14. What is the derivative of $f(x) = e^{\sin 3x}$?
Answer: First compute: $\frac{d}{dx}[\sin 3x] = 3\cos 3x$, by the chain rule. Then by the chain rule again,

$$f'(x) = e^{\sin 3x}\frac{d}{dx}[\sin 3x] = 3e^{\sin 3x}\cos 3x.$$

15. What is the derivative of $f(x) = (\ln x)^3$?
Answer: Use the chain rule: $f'(x) = 3(\ln x)^2/x$.

16. What is the derivative of $f(x) = \log_3 x$?
Answer: For this question, I just want you to know the formula and use it: $f'(x) = \frac{1}{x\ln 3}$.

17. What is the derivative of $f(x) = 2^{(x^5)}$?
Answer: use the chain rule and formula for derivatives of logarithmic functions. $f'(x) = 5 \ln 2 \cdot 2^{(x^5)} x^4$.

18. Find the derivative of a function $y = f(x)$ defined implicitly by $x^2 + 4y^2 = 4$.
Answer:

$$\begin{aligned} \frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) &= 0 \\ 2x + 8y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= \frac{-x}{4y}. \end{aligned}$$

19. What is the derivative of $x^{\sin 3x}$? HINT: Use logarithmic differentiation.
Answer:

$$\begin{aligned} y &= x^{\sin 3x} \\ \ln y &= \sin 3x \ln x \\ \frac{1}{y}\frac{dy}{dx} &= 3 \ln x \cos 3x + \frac{\sin 3x}{x}. \end{aligned}$$

Multiplying both sides by y and plugging in $y = x^{\sin 3x}$, we find that

$$\frac{dy}{dx} = x^{\sin 3x}(3 \ln x \cos 3x + \frac{\sin 3x}{x}).$$

20. What is the derivative of $(\sin x)^{\cos x}$?

Answer: The solution strategy is similar to the last problem:

$$\begin{aligned} y &= (\sin x)^{\cos x} \\ \ln y &= \cos x \ln(\sin x) \\ \frac{1}{y}\frac{dy}{dx} &= -\sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x} \\ &= \cos x \cot x - \sin x \ln(\sin x). \end{aligned}$$

Multiplying both sides by y and plugging in $(\sin x)^{\cos x}$, we find that

$$\frac{dy}{dx} = (\sin x)^{\cos x}(\cos x \cot x - \sin x \ln(\sin x)).$$

21. Suppose the distance in miles traveled by a car after t hours is given by $r(t) = 3t + \sqrt{t}$. Find expressions for the velocity $v(t)$ and acceleration $a(t)$ of the car at time t.

Answer:

$v(t) = r'(t) = 3 + \frac{1}{2\sqrt{t}}$.

$a(t) = r''(t) = v'(t) = -\frac{1}{4t\sqrt{t}}$.

22. Suppose I drop a ball from the top of a 200m tower. What is the velocity of the ball just before it hits the ground? What is the acceleration? You may assume that the distance $r(t)$ in meters that the ball has dropped after t seconds is given by $r(t) = 4.9t^2$.

Answer: The time t_f at which the ball hits the ground is the positive solution to $200 = 4.9t^2$. So $t_f = \frac{10\sqrt{2}}{\sqrt{4.9}}$.

$v(t) = 9.8t\ m/s$, so $a(t) = 9.8\ m/s^2$.

Thus $v(t_f) = 20\sqrt{9.8}\ m/s$ and $a(t_f) = 9.8\ m/s^2$.