TMAT/AMAT 118 Partial Practice Final

Name: ____________________________

Note: As a reminder, the actual final is cumulative, and will also cover material from the first two midterms. This partial practice final only covers material from after the cutoff for the second midterm.

1. The area of a circular disc is expanding at a rate of $10m^2/s$. What is the rate of change of the radius of the disc when the area of the disc is 400π?

Answer: This is a related rate problem, as considered in 3.9. Let A and r denote the area and radius of the disc, respectively. We are asked to find $\frac{dr}{dt}$ when $A = 400\pi$. $A = \pi r^2$, so when $A = 400\pi$, $r = 20$; thus the goal is to find $\frac{dr}{dt}$ when $r = 20$, which we write as $\frac{dr}{dt}|_{r=20}$. By the chain rule, $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$. By the power rule, $\frac{dA}{dr} = 2\pi r$. So we have $10 = 2\pi 20 \frac{dr}{dt}|_{r=20}$. Thus $\frac{dr}{dt}|_{r=20} = \frac{1}{4\pi} m/s$.

2. Let $y(t)$ denote the population of squirrels in Albany, NY t years after the year 2000. Suppose that were 1000 squirrels in Albany in 2000, and that $y(t)$ satisfies $y' = 1.01y$. How many squirrels will there be in Albany in 2019?

Answer: In general, the solution to $y' = ky$ for any constant k is Ce^{kt}, where $C = y(0)$. Plugging this in, we see that the answer is $1000e^{1.01(19)} = 1000e^{19.019}$.

3. Give the definition of a local minimum of a function f.

Answer: $f(c)$ is a local minimum of f if for every x near c, $f(x) \geq f(c)$.
Note 1: "near c" is understood to mean "in some open interval containing c."
Note 2: In the definition, it is understood that if f has a local minimum at c, then f is defined in an open interval containing c.

4. If the domain of $f(x)$ is $[0, \infty)$, is it possible for f to have a global maximum at $x = 0$? What about a local maximum?

Answer: Yes, f can have a global maximum at $x = 0$. For example, consider $f : [0, \infty)$ given by $f(x) = -x$. f cannot have a local maximum at $x = 0$, because f is only defined on one side of $x = 0$. (See Note 2 in the above problem.)

5. If the domain of f is $\mathbb{R}$ and the values of the local maxima of f are $0, 1$, and 2, what can be said about a global maximum of f?

Answer: f doesn't necessarily have a local maximum, but if it does, it is 2. This is because any global maximum that is defined in the interior of the domain of f (i.e., away from the endpoints of f) is also a local maximum.

6. True or false: Every function $f : [-1, 1] \to \mathbb{R}$ has an absolute maximum.

Answer: False. For example, take $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$ But if f is continuous, then according to the extreme value theorem, f does have an absolute maximum.

7. Find all critical numbers of each of the following functions.

- $f(x) = |x|$, **Answer**: $f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$ $f'(x)$ is undefined at $x = 0$. So the only critical number is 0.
- $f(x) = \sqrt[3]{x+1}$, (Hint: Use the chain rule to compute the derivative of f), **Answer**: $f'(x) = \frac{1}{(x+1)^{2/3}}$. $f'(x)$ is undefined at $x = -1$; it is defined and non-zero everywhere else. Thus the only critical number is -1.
- $f(x) = \frac{1}{3}x^3 - 4x + 7$. **Answer**: $f'(x) = x^2 - 4$, so the critical numbers are -2 and 2.

8. Let $f : [-1, 5] \to \mathbb{R}$ be given by $f(x) = \frac{1}{3}x^3 - 4x + 7$.

- On what interval(s) is f increasing? On what interval(s) is f decreasing? **Answer**: The computation in the third part of the last problem shows that $f'(x)$ is 0 only when $x = 2$. (Remember that the domain of f is $[-1, 5]$, so -2 is not a critical number for f.) By considering the graph of $f'(x)$, it is clear that $f'(x) > 0$ when $x \in (2, 5]$, and $f'(x) < 0$ for $x \in [-1, 2)$. Thus, by the increasing/decreasing test, f is increasing for $x \in (2, 5]$, and f is decreasing for $x \in [-1, 2)$.
- On what interval(s) is f concave up? On what interval(s) is f concave down? **Answer**: $f''(x) = 2x$. Thus, by the concavity test, f is concave down on $[-1, 0)$ and concave up on $(0, 5]$.
- At which x does f have a point of inflection? **Answer**: $x = 0$.
- What are the absolute max and absolute min of f? At what values of x are these attained? **Answer**: We solve this problem using the "closed interval method" (See page 281 of Stewart). $f(2) = 5/3$, $f(-1) = 10\frac{2}{3}$, $f(5) = \frac{125}{3} - 20 + 7 = 28\frac{2}{3}$. Thus the global max is $28\frac{2}{3}$, attained at $x = 5$, and the global min is $\frac{5}{3}$, attained at $x = 2$.
- What are local maxes and mins of f? At what values of x are these attained? **Answer**: Since f' exists everywhere, Fermat's Theorem tells us that any local max/min must be a critical number. The only critical number is $f(2)$. Since f has a global min at $f(2)$, it must also have a local min here. (Alternatively, the 2nd derivative test tells us that f has a local min at $x = 2$.) Thus, $\frac{5}{3}$ is the unique local min of f, and it is attained at $x = 2$.

9. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f(1) = 1$. Use the mean value theorem to complete the following: There exists $c \in (0, 1)$ such that _______.
Answer: $f'(c) = 1$.

10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f'(x) \leq -1$ for all $x \in (0, 1)$. What is the maximum value that $f(1)$ can possibly take? Hint: apply the mean value theorem to f on the interval $[0, 1]$.

Answer: By the mean value theorem, there exists $c \in (0,1)$ with $f'(c) = \frac{f(1)-f(0)}{1-0} = f(1)$. $f'(c) \leq -1$ so $f(1) \leq -1$. Thus, the maximum value $f(1)$ can possibly take is -1.

11. Apply l'Hospital's rule to compute $\lim_{x \to 0} \frac{x^2}{\ln(x+1)}$.

Answer: $\lim_{x \to 0} \frac{x^2}{\ln(x+1)} = \lim_{x \to 0} \frac{2x}{1} = \lim_{x \to 0} 2x = 0$.

12. Find the most general antiderivative of $\frac{1}{\sqrt{x}} + 2\cos(x) + 5 + 4^x$.

Answer: $2\sqrt{x} + 2\sin(x) + 5x + \frac{4^x}{\ln 4} + C$.

13. Solve the differential equation $f''(x) = 2x - \sin(x) + 1$, $f'(0) = 1$, $f(0) = 1$.

Answer: The most general antiderivative of $f''(x)$ is

$$f'(x) = x^2 + \cos(x) + x + C.$$

$f'(0) = 1 + C = 1$, so $C = 0$. Hence the most general antiderivative of $f'(x)$ is

$$f(x) = \frac{1}{3}x^3 + \sin(x) + \frac{1}{2}x^2 + D.$$

$f(0) = D = 1$, so the solution is

$$f(x) = \frac{1}{3}x^3 + \sin(x) + \frac{1}{2}x^2 + 1.$$