## AMAT/TMAT 118

## Notes and Exercises on Sets and Functions

This set of notes covers more or less the same material that we covered in class on sets and functions. There is also optional material which I will not cover in class, but which is relevant. I encourage you to read this.

## 1 Introductory Remarks

The language of sets and functions is essential to the foundations of mathematics; nearly any object you encounter in your mathematical studies can be defined in set-theoretic terms. Higher level mathematics can get confusing quickly, even for specialists. Often times, being clear on the precise definitions of the sets and functions one is working with is a first step towards understanding what is happening.

My treatment of set theory will sometimes be slightly informal. The goal is to acquaint you with the key concepts and definitions.

## 2 Definition of a Set

What is a set? Here is the informal definition I gave in class:
Definition 1. A set is a collection of distinct elements.
This is somehow, not a very satisfying definition, because we haven't defined elements. In an advanced class on set theory, you would see a more careful description of a set, which avoids this fuzzy use of the term elements (but arguably just moves this fuzziness somewhere else).

Definition 1 is best made clear by some examples.
Example 2.1. $S_{1}=\{A, B\}$ is a set. This is the set consisting of the two letters $A$ and $B$. As shown here, we can specify a set by listing its elements. When we do that, we put the elements in curly braces. The order in which we write the elements in the curly braces doesn't matter.

Example 2.2. $S_{2}=\{0,1,2\}$ is a set, consisting of the three numbers 0,1 , and 2.

Example 2.3. $S_{3}=\{ \}$ is a set. This is called the empty set, and is also written as $\emptyset$. This is the unique set with no elements.

Example 2.4. We can have a set whose elements are themselves sets. For example for sets $S_{1}, S_{2}$, and $S_{3}$ defined as above, $S_{4}=\left\{S_{1}, S_{2}, S_{3}\right\}$ is a set.

Example 2.5. $S_{5}=\{0,1, A, B\}$ is a set. Thus, we see that a set is allowed to have elements of different types.

Example 2.6. Infinite sets are allowed. In particular the natural numbers

$$
\{0,1,2,3,4, \ldots\}
$$

form a set, denoted $\mathbb{N}$. Similarly, the integers

$$
\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}
$$

also form a set, denoted $\mathbb{Z}$.
Example 2.7. Sets with a single element are allowed. These are called singleton sets. For example, $\{0\}$ is a set with one element. Note that this is not the same as the number 0 ; rather it is the singleton set with element 0 . More confusingly, $\{\{0\}\}$ is also a set, namely the singleton set with element $\{0\}$.

Notation for an Element Contained in a Set If a set $S$ contains an element $a$, we write $a \in S$. Otherwise, we write $a \notin S$.

Example 2.8. $-1 \notin \mathbb{N}$, but $-1 \in \mathbb{Z}$.
Exercise 1. Is it true that $0 \in\{\{0\}\}$ ?

Equality of Sets We say sets $S$ and $T$ equal, and write $S=T$, if the elements of $S$ and $T$ are the exactly same. If $S$ and $T$ are not equal, we write $S \neq T$.

Example 2.9. If $S=\{a, b, c\}$ and $T=\{A, B, C\}$, then $S \neq T$.
Example 2.10. If $S=\{a, b, c\}$ and $T=\{c, b, a\}$, then $S=T$.
Example 2.11. If $S=\{0\}$ and $T=\{\{0\}\}$, then $S \neq T$.
Exercise 2. If $S=\{\{a\},\{b\}\}$, and $T=\{\{a\}, b\}$, is it true that $S=T$ ?

## 3 Operations on Sets

We'll be primarily interested in four operations on sets: union, intersection, taking subsets, and cartesian products.

Union The union of two sets $S$ and $T$, denoted $S \cup T$, is the set consisting of all elements in either $S$ or $T$.

Example 3.1. If $S=\{A, B\}$ and $T=\{B, C\}$, then $S \cup T=\{A, B, C\}$.
Exercise 3. If $S=\{A, B\}$, what is $S \cup \emptyset$ ?
Exercise 4. For $S$ as in the previous exercise, what is $S \cup S$ ?
Union The intersection of two sets $S$ and $T$, denoted $S \cap T$, is the set consisting of all elements in both $S$ and $T$.

Example 3.2. If $S=\{A, B, C\}$ and $T=\{B, C, D\}$, then $S \cap T=\{B, C\}$.
Exercise 5. If $S=\{A, B\}$, what is $S \cap \emptyset$ ?
Exercise 6. If $S=\{A, B\}$, what is $S \cap S$ ?
Subsets For sets $S$ and $T$, we say $S$ is a subset of $T$, and write $S \subset T$, if every element of $S$ is also an element of $T$.

Example 3.3. $\{A, B\} \subset\{A, B, C\}$, but $\{A, B, C\} \not \subset\{A, B\}$.
Example 3.4. For any set $S$, we have that $S \subset S$. Morever, $\emptyset \subset S$. (As noted in class, the fact that the statement "every element of $\emptyset$ is an element of $S$ " is true may seem a little strange, since the empty has no elements, but by convention, this is interpreted this to be true.)

Exercise 7. Is it true that if $S \subset T$ and $T \subset S$, then $S=T$ ?
Example 3.5. Let us write the set of all subsets of $\{A, B\}$. This is

$$
\{\{A\},\{B\},\{A, B\},\{ \}\} .
$$

Note that $A$ is not a subset of $B$, but $A \in\{A, B\}$. Conversely, $\{A\} \subset\{A, B\}$, but $\{A\} \notin\{A, B\}$.

Exercise 8. Find all the subsets of $\{\{A, B\},\{ \}\}$.

Specifying a Subset Via a Property We sometimes specify a subset via a property its elements satisfy. For example, the set of even numbers, sometimes denoted $2 \mathbb{Z}$, can be specified as the set of integers $z$ such that $z$ is divisible by 2 . We write this in symbols as follows:

$$
2 \mathbb{Z}=\{z \in \mathbb{Z} \mid z \text { is divisible by } 2\} .
$$

the symbol "|" translates into words as "such that."
Cartesian Products For sets $S$ and $T$, the Cartesian Product of $S$ and $T$, denoted $S \times T$, is the set of all ordered pairs $(s, t)$ with $s \in S$ and $t \in T$. In symbols, we write this as follows:

$$
S \times T=\{(s, t) \mid s \in S, t \in T\}
$$

Example 3.6. For $S=\{1,2\}$ and $T=\{A, B\}$,

$$
S \times T=\{(1, A),(1, B),(2, A),(2, B)\}
$$

Example 3.7. For $S=\{1,2\}$ and $T=\emptyset, S \times T=\emptyset$.
Example 3.8. By definition, $\mathbb{Z} \times \mathbb{Z}$ is the said of all ordered pairs of integers.
Exercise 9. For $S=\{1,2\}$, what is $S \times S$ ?

## 4 Functions

Definition 2. Given sets $S$ and $T$, a function from $S$ to $T$ is a rule which assigns each $s \in S$ exactly one element in $T$. This element is denoted $\mathrm{f}(\mathrm{s})$. We call $S$ the domain of $f$, and $T$ the codomain of $T$. (Note that we will not call $T$ the range; as we will see below, the range is something different.).

We write the function as $f: S \rightarrow T$.
For intuition, it can be helpful to think of a function $f: S \rightarrow T$ as a machine which takes as input an element of $s \in S$, and returns an element $f(s) \in T$.

Example 4.1. We can specify a function by explicitly specifying its action on each element of the domain. For example, Let $S=\{1,2\}$ and $T=\{A, B\}$. We can define a function $f: S \rightarrow T$ by

$$
f(1)=A, f(2)=B
$$

We can define a different function $g: S \rightarrow T$ by

$$
g(1)=A, g(2)=A
$$

Example 4.2. We can sometimes also specify a function via a formula. For example, we can define a function $h: \mathbb{Z} \rightarrow \mathbb{Z}$ by the formula $h(z)=z^{2}$.

Range of a Function Given a function $f: S \rightarrow T$, we define the range of $F$ to be the subset of $T$ given by

$$
\text { range } f=\{t \in T \mid t=f(s) \text { for some } s \in S\}
$$

Informally, range $f$ is the set of elements which are "hit by $f$."
Example 4.3. For $f$ and $g$ the functions in Example 4.1, range $f=\{A, B\}$, and range $g=\{A\}$. For $h$ as in Example 4.2,

$$
\text { range } h=\{0,1,4,16,25,36,49, \ldots\} .
$$

In other words, range $h$ is the set of perfect squares.

Inclusion and Identity Functions For any set $S$, the identity function on $S$ is the function Id : $S \rightarrow S$, given by $\operatorname{Id}(s)=s$ for all $s \in S$. We sometimes also write the identity function on $S$ as $\operatorname{Id}_{S}$.

Example 4.4. For $S=\{A, B\}$, Id $: S \rightarrow S$ is given by

$$
\operatorname{Id}(A)=A, \operatorname{Id}(B)=B
$$

Exercise 10. What is the range of the identity function Id : S $\rightarrow$ ?
For any set $T$ with subset $S$, the inclusion function $j: S \rightarrow T$ is the function given by $j(s)=s$ for all $S$.
Example 4.5. For $S=\{A, B\}, T=\{A, B, C\}$, the inclusion $j: S \rightarrow T$ is given by

$$
j(A)=A, j(B)=B
$$

Example 4.6. For any sets $S$ and $T$, the range of the inclusion function $j: S \rightarrow T$ is just $S$.

Exercise 11. For $S=\{A, B\}, T=\{A, B, C\}$, how many functions $f: S \rightarrow$ $T$ are there with the range of $f$ equal to the range of the inclusion $j: S \rightarrow T$ ? List them.

Arithmetic Operations as Functions We have promised that the language of sets and functions serves as a foundation for mathematics. If that is true, it ought to be the case that familiar arithmetic operations on integers can be expressed in terms of sets and functions.

Indeed this is the case. For example,

- Addition of integers can be interpreted as a function $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ sending $(x, y)$ to $x+y$.
- Multiplication of integers can be interpreted as a function $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ sending $(x, y)$ to $x y$.

Equation-Solving as Finding a Subset (Optional) Equation solving can be interpreted in set-theoretic terms. Consider, for example, the following high school (or middle school) math problem.

$$
\text { Solve } x+3=0 \text { for } x .
$$

We define a function $f(x): \mathbb{Z} \rightarrow \mathbb{Z}$ by $f(x)=x+3$. Then solving the above equation is equivalent to finding the set $\{x \in \mathbb{Z} \mid f(x)=0\}$.

Exercise 12 (bonus). The example above is a bit special because the right hand side of the equation is 0 . Generalizing the above, explain how the solution to the equation $f(x)=g(x)$ can be interpreted as a subset of $\mathbb{Z}$ for any functions $f: \mathbb{Z} \rightarrow \mathbb{Z}, g: \mathbb{Z} \rightarrow \mathbb{Z}$. (This is not deep, but it is good to think about this once.)

Composition of functions Given functions $f: S \rightarrow T$ and $g: T \rightarrow U$, we say that the composition of $f$ and $g$ is the function $g \circ f: S \rightarrow U$ given by $g \circ f(x)=g(f(x))$.

Example 4.7. Let $S=\{1,2\}, T=\{A, B\}, U=\{x, y\}$. Let $f: S \rightarrow T$ be given by $f(1)=f(2)=A$, and let $g: T \rightarrow U$ be given by $g(A)=x$, $g(B)=y$. Then $g \circ f$ is given by $g(1)=g(2)=x$.

## 5 1-1, Onto, and Bijective Functions

The following three definitions are very important in the study of functions.
Let $f: S \rightarrow T$ be a function.

- $f$ is said to be $1-1$ if whenever $s \neq t \in S$, we have $f(s) \neq f(t)$. In other words, a 1-1 function is one which always maps distinct elements to distinct elements.
- $f$ is said to be onto if range $f=T$.
- $f$ is said to be a bijection if it is both 1-1 and onto.


## Example 5.1.

- The function $f$ of Example 4.1 is a bijection.
- The function $g$ from the same example is not 1-1 because $g(1)=g(2)=$ A. $g$ is also not onto because $B \notin$ range $g$.
- The function $h$ from Example 4.2 is not 1-1, because $x$ and $-x$ always have the same square. $h$ is not onto because many numbers are not perfect squares.

Example 5.2. For any set $S$, the identity map is a bijection. For $S$ any subset of a set $T$, the inclusion $j: S \rightarrow T$ is always 1-1.

Exercise 13. Let $S=\{1,2\}$ and $T=\{A, B, C\}$.
(i) How many functions are there from $S$ to $T$ ?
(ii) How many 1-1 functions are there from $S$ to $T$ ?
(iii) How many onto functions are there from $S$ to $T$ ?
(iv) How many bijections are there from $S$ to $T$ ?

Inverses of functions Functions $f: S \rightarrow T$ and $g: T \rightarrow S$ are inverses if $g \circ f=\mathrm{Id}_{\mathrm{S}}$ and $f \circ g=\mathrm{Id}_{T}$. We say $g$ is an inverse of $f$ (and vice versa).

## Important Fact 5.3.

(i) A map $f: S \rightarrow T$ is a bijection if and only if $f$ has an inverse.
(ii) If an inverse of $f$ exists, it is unique.

Proof of ( $i$ ). First, suppose that $f$ is a bijection. Then for each $t \in T$, there is exactly one $s \in S$ with $f(s)=t$. We define $g: T \rightarrow S$ by $g(t)=s$. It is very easy to check that $f$ and $g$ are inverses.

Conversely, if $f$ and $g$ are inverses then $f$ must be onto: The condition that $f \circ g(s)=s$ for all $s$, means that for all $s$, there is some $t \in T$ such that $f(t)=s$. Namely, $t=g(s)$. Similarly, $f$ must be 1-1: If not, then we have $s_{1} \neq s_{2}$ with $f\left(s_{1}\right)=f\left(s_{2}\right)$. But then $g \circ f\left(s_{1}\right)=g \circ f\left(s_{2}\right)$ which means that $g \circ f \neq \mathrm{Id}_{S}$. [NOTE: I am using that a statement is true if and only if the contrapositive of a statement is true. This is a basic fact of logic. If you don't understand this, let me know, or look it up.] Thus $f$ is a bijection. This proves (i). The proof of (ii) is left as an exercise.

Exercise 14 (bonus). Proof part (ii) of the important fact.
Exercise 15. Find the inverse of the function $f$ from Example 4.1. (HINT: See the first couple of sentences of the above proof.)

Exercise 16. For which integers $a$ does the function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x)=x+a$ have an inverse? For each such $a$, what is the inverse?

Exercise 17. For which integers $a$ does the function $f: \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(x)=a x$ have an inverse? For each such $a$, what is the inverse?

Restricting to the Range The following fact is worth keeping in mind: Given any function $f: S \rightarrow T$, we have a function

$$
\tilde{f}: S \rightarrow \text { range } f
$$

given by $\tilde{f}(s)=f(s)$. We call this function the restriction of $f$ to its range. Intuitively, $\tilde{f}$ is the function obtained by removing all the "extra stuff" in the codomain that is not hit by $f . \tilde{f}$ has the same domain and the same "rule" as $f$, just a smaller codomain.

Clearly, $\tilde{f}$ is onto. It's also easy to see that $\tilde{f}$ is $1-1$ if and only if $f$ is $1-1$. Therefore, if $f$ is $1-1, \tilde{f}$ is a bijection.

Example 5.4. Let $S=\{1,2\}$ and $T=\{A, B, C\}$. Let $f: S \rightarrow T$ be given by $f(1)=A$ and $f(2)=B . f$ is $1-1$, but it is not onto. The restriction of $f$ to its range is a map $\tilde{f}: S \rightarrow\{A, B\}$. $\tilde{f}$ has an inverse $g:\{A, B\} \rightarrow S$ given by $g(A)=1$, and $g(B)=2$.

### 5.1 Partition of a Set

We will use the notion of partitions of sets to talk about rational and real numbers. A partition $P$ of a set $S$ is a set such that:

- Elements of $P$ are non-empty subsets of $S$, called equivalence classes.
- Each element of $T$ is contained in exactly one element of $P$.

We write the equivalence class in $P$ of an element $s \in S$ as [s].
Example 5.5. $\{\{1\},\{2,3\}\}$ is a partition of $\{1,2,3\}$. $\{\{1,2,3\}\}$ is also a partition of $\{1,2,3\}$.
Exercise 18. Is $\{1,2,3\}$ a partition of $\{1,2,3\}$ ?
Exercise 19. Find all partitions of $\{1,2,3\}$.
Example 5.6. Letting $2 \mathbb{Z}+1 \subset \mathbb{Z}$ denote the set of odd integers, $\{2 \mathbb{Z}, 2 \mathbb{Z}+1\}$ is a partition of the $\mathbb{Z}$.

## 6 Rational Numbers

So far, I have avoided discussion of reals and rational numbers in this treatment of set theory. But for calculus, these are extremely important.

Important Note: I realize that this material on reals and rationals is a bit abstract, maybe a notch above what I want for this class in terms of abstraction. It also doesn't get used so often in the actual practice of of solving calculus problems, and so is often skipped in a first course.

Still, this is an honors course, and I think it's very valuable for students to have some idea of what the rationals and (especially) the reals really are as sets, so I decided to cover it. But in recognition of the difficulty of this part, I've decided not to assign homework on or test you on the set-theoretic aspects of the reals and rationals. I would, however, like you to read this once and take what you can from it.

To be clear, you *will* be tested on the other set theory stuff in these notes, and will be expected to have the standard high-school fluency with reals and rationals, as in the diagnostic quizzes.

Let's start with the rational numbers, denoted $\mathbb{Q}$. What exactly is $\mathbb{Q}$ as a set? In grade school, you've been told the following:

- A rational number (i.e. fraction) is an expression of the form $\frac{a}{b}$, where $a$ and $b$ are integers, with $b \neq 0$.
- Fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if they reduce to the same fraction, i.e. when $a d=b c$.

Now we want to make the connection between these ideas and set theory. To start, let's consider the set

$$
\tilde{\mathbb{Q}}=\left\{\left.\frac{a}{b} \right\rvert\,(a, b) \in \mathbb{Z} \times \mathbb{Z}, b \neq 0\right\} .
$$

That is, $\tilde{\mathbb{Q}}$ is the set whose elements are fractions. Though it is tempting to define the rational numbers to be $\tilde{\mathbb{Q}}$, this is not quite the right, because multiple elements of $\tilde{\mathbb{Q}}$ represent the number. For example, in $\tilde{\mathbb{Q}}, \frac{1}{2}$ and $\frac{2}{4}$ are different elements, but we want to think of them as the same.

The (cleanest) set-theoretic solution is to take $\mathbb{Q}$ to be a partition of $\tilde{\mathbb{Q}}$. In this partition, the equivalence class of $\frac{1}{2}$ is the set of all fractions representing the same number as $\frac{1}{2}$, i.e.,

$$
\left[\frac{1}{2}\right]=\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\right\} .
$$

More generally, we take

$$
\left[\frac{a}{b}\right]=\left\{\left.\frac{c}{d} \in \tilde{\mathbb{Q}} \right\rvert\, a d=b c\right\} .
$$

So set theoretically, we think of a rational number as the set of all the fractions representing it!

Regarding the Rationals as a Subset of the Integers As a general principle, when when we have a specified 1-1 map $j: S \rightarrow T$, we sometimes think of range $j$ as a copy of $S$. As an example, there is a 1-1 function $j: \mathbb{Z} \rightarrow \mathbb{Q}$ given by $j(z)=\left[\frac{z}{1}\right]$. We think of the range of this function as a copy of $\mathbb{Z}$ sitting inside of $\mathbb{Q}$. In this sense, $\mathbb{Z} \subset \mathbb{Q}$.

Arithmetic on Rationals As in the case of integers, addition and multiplication can be defined as functions $\mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$ :

- The formula for addition of fractions is $\left[\frac{a}{b}\right]+\left[\frac{c}{d}\right]=\left[\frac{a d+b c}{b d}\right]$.
- The formula for multiplication of fractions is $\left[\frac{a}{b}\right]\left[\frac{c}{d}\right]=\left[\frac{a c}{b d}\right]$.

Letting $\mathbb{Q} \backslash\{0\}$ denote the rational numbers, with 0 removed. We also have a division operation in $\mathbb{Q}$, which is a function $\mathbb{Q} \times(\mathbb{Q} \backslash\{0\}) \rightarrow \mathbb{Q}$, given by $\left[\frac{a}{b}\right] /\left[\frac{c}{d}\right]=\left[\frac{a}{b}\right]\left[\frac{d}{c}\right]=\left[\frac{a d}{b c}\right]$.

Rationals as Decimals Let $D$ denote the set of decimal numbers (possibly extending infinitely to the right). For example,

$$
.5 \in D, \quad .047 \in D, \quad 4.56 \overline{232323} \in D, \quad .01001000100001 \ldots \in D
$$

There is a 1-1 map $\mathbb{Q} \rightarrow D$ sending a rational number to its decimal representation. The map is given by long division. range $j$ is the set of all decimals which either terminate (like .047), or eventually repeat themselves infinitely often (like $4.56 \overline{232323}$ ). We think of range $j$ as a copy of $\mathbb{Q}$.

## 7 Real Numbers

In calculus, the basic objects of study are functions $f: U \rightarrow \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers and $U \subset \mathbb{R}$ is some subset of $\mathbb{R}$. For example, $U$ could be the interval $[0,1]$, or $U$ could be $\mathbb{R}$.

But what is the set $\mathbb{R}$, exactly? Before we go into the details, let us remind ourselves of the high level idea of the real numbers: $\mathbb{R}$ is a set with $\mathbb{Q} \subset \mathbb{R}$. Working with $\mathbb{R}$ instead of $\mathbb{Q}$ gives us some extra numbers which we really need in mathematics:

- We get all square roots, cube roots, fourth roots, etc, of positive rational numbers. We need these to define the length of the diagonal of a square or cube.
- We get $\pi$. We need this to define the circumference or area of a circle.
- We get $e$. We need this for the solution to the simplest differential equation $f^{\prime}=f$ to exist. (We'll talk about this later).

The real numbers are a single construction that gives us all these numbers and lots of others as well that are convenient to have around.

Real Numbers as Decimals In high school classes, real numbers are sometimes described as infinite decimal sequences, i.e., elements of the set $D$ defined above. For example, $\pi=3.141592654 \ldots$, where the sequence of digits goes on forever to the right. We will treat real numbers from the perspective of infinite decimals, though there are other equivalent ways o define the reals that have their advantages. (Other approaches more elegant but more abstract.)

One might hope to define $\mathbb{R}$ simply to be equal to $D$. This is almost right, but not quite. The issue is that we need to stipulate, for example, that $1.0=. \overline{999}$. More generally, we need to stipulate that any decimal ending in an infinite sequence of 9 's is equal to the decimal obtained by removing the infinite sequence of 9 's and making rightmost digit of the resulting decimal one larger.

How do we make this precise in set-theoretic terms? We already saw a similar problem when constructing $\mathbb{Q}$, where more than one fraction represented the same rational number. As with $\mathbb{Q}$, the solution here is to use a partition:

We define $\mathbb{R}$ to be the partition of $D$ such that:
1 Each equivalence class in the partition has either one or two elements.
2 The equivalence classes with two elements are the pairs $\{x, y\}$ given as follows: $x$ is a decimal ending in an infinite sequence of 9 's; $y$ is the decimal obtained by removing the infinite sequence of 9 's from $x$ and making rightmost digit of the resulting decimal one larger.

For example the real number 1 is represented by the two-element equivalence class $\{1.0, . \overline{999}\}$, and the real number $\frac{1}{3}$ is represented by the singleton equivalence class $\{. \overline{333}\}$.

Arithmetic on Reals To define arithmetic (e.g., addition, multiplication, and division) on $\mathbb{R}$ one has to to think about what it means to do arithmetic on infinite decimal sequences, and one has to check that the the definitions play nicely with the equivalence classes. This all works out as it should. I will not go into the details.

Final Remarks on Reals In practice, when working with $\mathbb{R}$ we never actually carry out arithmetic on infinite decimal sequences, and don't worry
often about the idea that real numbers are equivalence classes. Instead, we treat real numbers symbols, subject to certain rules of manipulation, as you have throughout high school. For example,

$$
\frac{e}{1-3 \sqrt{\pi}}
$$

can be represented as an infinite decimal sequence, but we are happy to just work with it symbolically.

## 8 Graph of a Function (Not Covered In Class But Required)

The graph of a function $f: S \rightarrow T$ has a formal set-theoretic definition. It is defined as the subset of $S \times T$ given by

$$
\{(s, f(s) \mid s \in S\}
$$

Example 8.1. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(r)=|r|$. Then the graph of $f$ is the set $\{(r,|r|) \mid r \in \mathbb{R}\}$. This can be plotted as a curve in the plane.

Remark 8.2. If we plot the graph of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ in the plane, then the intersection of the graph of $f$ with any vertical line has exactly one point. But the intersection of the graph with any horizontal line may contain several points, or no points at all.

If the graph of $f$ intersects each horizontal line at most once, $f$ is $1-1$. If the graph of $f$ intersects each horizontal line at least once, $f$ is onto. Make sure you understand why these statements are true!

## 9 Notation Index

- $\{x, y, z\}=$ the set containing the elements $x, y$, and $z$.
- $\mathbb{N}=$ natural numbers, i.e., $\mathbb{N}=\{0,1,2,3, \ldots\}$,
- $\mathbb{Z}=$ integers, i.e., $\mathbb{Z}=\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$,
- $\mathbb{Q}=$ rational numbers,
- $\mathbb{R}=$ real numbers.
- $a \in S$ means " $a$ is an element of the set $S$. .
- $a \notin S$ means " $a$ is an element of the set $S$."
- $\emptyset=$ the empty set, i.e., the set with no elements. This is also can be written as $\}$.
- $S \subset T$ means " $S$ is a subset of $T$ ",
- $S \not \subset T$ means " $S$ is not a subset of $T$ ".
- $S \cup T=$ the union of sets $S$ and $T$,
- $S \cap T=$ the intersection of sets $S$ and $T$,
- $S \times T=$ the Cartesian product of sets $S$ and $T$.
- Elements of $S \times T$ are denoted $(s, t)$, for $s \in S$ and $t \in T$. (Note that this notation conflicts with the notation for open intervals in $\mathbb{R}$, but that's the convention.)
- $\mid=$ "such that." For example, $\{y \in \mathbb{Z} \mid y$ is prime $\}$ means "the set of all integers $y$ such that $y$ is prime."
- $f: S \rightarrow T$ means $f$ is a function with domain $S$ and codomain $T$.
- For $f: S \rightarrow T$ and $U \subset S, f(U)$ denotes the image of $U$ under $f$. (The definition of this appears in the next set of notes, not these.)

