AMAT/TMAT 118

THE PRECISE DEFINITION OF A LIMIT

Stewart gives the precise definition of a limit in Section 2.4, using a traditional ϵ-δ formulation. This is a good way of giving the definition, and has its advantages. In class, I gave an alternative (but equivalent) version of the definition. My version requires a bit of an investment in language to state. But I feel that once the language is established, this definition expresses the geometric intuition behind limits more transparently. In these notes, I present both definitions.

1 Preliminary Definitions

Images of Sets Under Functions Given $f : S \to T$ any function and $U \subset S$, define $f(U) \subset T$ by

$$f(U) = \{t \in T \mid t = f(s) \text{ for some } s \in U\}.$$

Intuitively, $f(U)$ is the subset of T consisting of all elements that are hit by elements of U. Note that $\operatorname{range} f = f(S)$, so this definition generalizes the definition of range to arbitrary subset of U.

Exercise 1. Let $S = \{A, B, C\}$, $T = \{X, Y, Z\}$, and $f : S \to T$ be given by $f(A) = X$, $f(B) = Y$, $f(C) = X$. Let $U = \{A, B\}$. What is $f(U)$?

Answer: $f(U) = \{X, Y\}$.

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let I be the interval $[-1, 1]$, J be the interval $(-1, 1)$, and K be the interval $(0, \infty)$. Express each of $f(I)$, $f(J)$, $f(K)$, and $f(\mathbb{R})$ in interval form.

Answer: $f(I) = [0, 1]$, $f(J) = [0, 1)$, $f(K) = (0, \infty)$, $f(\mathbb{R}) = (0, \infty)$.

Balls and Punctured Balls For $p \in \mathbb{R}$, a *ball centered around a point* p is simply an interval of the form $(p - \epsilon, p + \epsilon)$ for some $\epsilon > 0$.

For $p \in \mathbb{R}$, a *punctured ball centered around* p is a set of the form

$$(p - \epsilon, p + \epsilon) \setminus \{p\}$$

for some $\epsilon > 0$. Note that we can write this set in a few different ways:

$$\begin{aligned}
&(p-\epsilon, p+\epsilon) \setminus \{p\} \\
&= (p-\epsilon, p) \cup (p, p+\epsilon) \\
&= \{x \in \mathbb{R} \mid 0 < |x-p| < \epsilon\}.
\end{aligned}$$

Exercise 3. Is $(0,1)$ a ball centered around any point? If so, what point?

Exercise 4. Is $[0,1]$ a ball centered around any point? If so, what point?

2 Practice with "for every ... there exists ..." Expressions

As we will see below, the precise definition of a limit uses language of the form "for every ... there exists ...." Coupled with the other new ideas appearing in the definition, this kind of language might be confusing for some. So before giving the precise definition of a limit, we'll practice a bit with this language, via some examples and exercises.

Note: We covered most but not all of these in class.

Example 2.1. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. For every $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ with $b > a$. For example, we can take $b = a + 1$. Or we could take $b = a + 2$; for each a, are many choices for b.

Exercise 5. Is it true that for every $a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ with $b < a$?

Example 2.2. Is it true that for every $a \in [0,1]$, there exists $b \in [0,1]$ with $b > a$? No, because for $a = 1$, there is no $b \in [0,1]$ with $b > a$.

Exercise 6. Is it true that for every $a \in (0,1)$, there exists $b \in (0,1)$ with $b > a$?

Exercise 7. Is it true that for every $y \in \mathbb{Z}$, there exists $z \in \mathbb{Z}$ with $z = -y$?

Exercise 8. Is it true that for every $y \in \mathbb{Z}$, there exists $z \in \mathbb{Z}$ with $z = \frac{1}{y}$?

Example 2.3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Is it true that for every $y \in \mathbb{R}$, there exists x with $f(x) > y$? Yes. For example, if $y < 1$, we can take $x = 1$. If $y > 1$, we can take $x = y$. (To give a more intuitive but less precise explanation, the answer is yes because we can make x^2 arbitrarily large by taking x to be large.)

Exercise 9. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = -x^2$. Is it true that for every $y \in \mathbb{R}$, there exists x with $f(x) > y$?

Exercise 10. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$. Is it true that for every set $T \subset \mathbb{R}$, there exists a set $S \subset \mathbb{R}$ with $T \subset f(S)$ and $S \neq \mathbb{R}$?

Exercise 11. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Is it true that for every set $T \subset \mathbb{R}$, there exists a set $S \subset \mathbb{R}$ with $T \subset f(S)$?

3 Limits

Definition 1 (Limit)**.** Suppose we are given:

- $S \subset \mathbb{R}$,
- $p, L \in \mathbb{R}$ such that S contains a punctured ball centered at p,
- a function $f : S \to \mathbb{R}$.

We write

$$\lim_{x \to p} f(x) = L$$

if for every ball C centered at L, there exists a punctured ball $B \subset S$ centered at p with $f(B) \subset C$.

Like many things in math, this definition is best understood with a picture, like the one I showed in class.

Remark 3.1. Note that in the above definition, we do not require $p \in S$. But we *do* require that S contains a punctured neighborhood of p.

Remark 3.2. It can be proven that if there exists $L \in \mathbb{R}$ with $\lim_{x \to p} f(x) = L$, then such L is unique.

Traditional definition of a limit The traditional $\epsilon - \delta$ definition of a limit is as follows:

Definition 2 (ϵ-δ definition of a limit)**.** Suppose we are given:

- $S \subset \mathbb{R}$,

- $p, L \in \mathbb{R}$ such that S contains a punctured ball centered at p,
- a function $f : S \to \mathbb{R}$,

we write

$$\lim_{x \to p} f(x) = L$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if} \quad 0 < |x - p| < \delta \quad \text{then} \quad |f(x) - L| \leq \epsilon.$$

Remark 3.3. You should take some some time to think about why this definition is equivalent to the one above. Hint: The statement

$$\text{if} \quad 0 < |x - p| < \delta \quad \text{then} \quad |f(x) - L| \leq \epsilon.$$

is equivalent to the statement

for B the punctured ball $(p - \delta, p + \delta) \setminus \{p\}$
and C the ball $(L - \epsilon, L + \epsilon)$, we have $f(B) \subset C$.

Example 3.4. Consider $f : \mathbb{R} \setminus \{0\} \to R$, given by $f(x) = x$. We explain why

$$\lim_{x \to 1} f(x) = 1,$$

using the precise definition of a limit. For every ball C centered at $L = 1$, C is of the form $C = (1 - \epsilon, 1 + \epsilon)$ for some $\epsilon > 0$. We take $B = C \setminus \{1\}$. This is a punctured ball centered at $p = 1$. Given how we defined f, it is clear that $f(B) = B \subset C$. This shows that that

$$\lim_{x \to 1} f(x) = 1.$$

Example 3.5. Consider $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} x & x \leq 1 \\ x + 1 & \text{otherwise} \end{cases}.$$

We explain why

$$\lim_{x \to 1} f(x)$$

does not exist, using the precise definition of a limit. For B any punctured ball centered around 1, we can write $B = (1-\delta, 1) \cup (1, 1+\delta)$, for some $\delta > 0$. Then $f(B) = (1-\delta, 1) \cup (2, 2+\delta)$. For any L in $\mathbb{R}$ and $C = (L-1/2, L+1/2)$, we cannot have $f(B) \subset C$, since $f(B)$ contains a pair of points more than distance 1 apart. This shows that

$$\lim_{x \to 1} f(x) \neq L.$$

Since this is true for any $L \in \mathbb{R}$, the limit in question does not exist, as claimed.