Math 4242 Sec 40 Quiz 7 Solutions

Name: ______________________

1. (3 points) For $A = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & -2 & 6 \\ -2 & 2 & 2 & -10 \\ 0 & -2 & 0 & 4 \end{pmatrix}$, is the set of columns of A linearly independent? Show that your answer is correct.

Answer: No: We reduce A to row echelon form to obtain the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix has a non-basic column, so $N(A) \neq \{\vec{0}\}$. Hence, the set of columns of A is linearly dependent.

2. (3 points) For A as in problem 1, give two different bases B_1, B_2 for $\mathcal{R}(A)$, the range of A. Explain why these are in fact bases.

Hint: you may find it helpful to use fact that performing row operations on a matrix preserves any linear dependencies amongst the columns; this was shown in Meyer Chapter 3 (top of page 136), and discussed in class a while ago.

Answer: Usually, a vector space has many bases, and $\mathcal{R}(A)$ is no exception. So there are many correct answers to this problem. Here's one answer:

As explained in Meyer, the basic columns of A form a basis for $\mathcal{R}(A)$. Hence, by problem 1), $S = \{A_{*1}, A_{*2}, A_{*3}\}$ is a basis for $\mathcal{R}(A)$.

Another basis for $\mathcal{R}(A)$ is given, for example, by $S' = \{2A_{*1}, A_{*2}, A_{*3}\}$.

$$spn(S') = \text{Span(S)} = \mathcal{R}\text{(A)}$$

because $S' \subset \text{Span(S)}$ and $S \subset \text{Span(S}')$. Furthermore S' is linearly independent because if

$$r_1(2A_{*1}) + r_2A_{*2} + r_3A_{*3} = (2r_1)A_{*1} + r_2A_{*2} + r_3A_{*3} = \vec{0},$$

then by the linear independence of S, $r_1 = r_2 = r_3 = 0$.

For a slightly more interesting answer, yet another basis is given by $S'' = \{A_{*1}, A_{*2}, A_{*4}\}$. By considering the reduced echelon form of A, and appealing to the hint, we can see that S'' is linearly independent and that $\text{Span(S}'') = \text{Span(S}')$, so S'' is in fact a basis.

IMPORTANT NOTE: It is usually not true that for B a row echelon form of A, the basic columns of B themselves form a basis for $\mathcal{R}(A)$: Usually $\mathcal{R}(\mathcal{A})$ and $\mathcal{R}(\mathcal{B})$ will be different. (Can you find a simple example which demonstrates this?)

3. (2 points) For $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$, write down all subsets of S which are bases for $\mathbb{R}^2$.

Answer:
$\dim \mathbb{R}^2 = 2$, so by Proposition 3.1 from (the revised version of) my notes, any linearly independent subset $Q \subset S$ with $|Q| = 2$ is a basis for $\mathbb{R}^2$. Hence every 2-element subset of S except $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$.

4. (2 points) Let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis for a vector space V. Prove that each $v \in V$ can be expressed as a linear combination of the b_i's:

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$$

in only one way–i.e., the coordinates α_i are unique.

Answer: If

$$v = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n = \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_n b_n$$

then

$$(\alpha_1 - \beta_1) b_1 + (\alpha_2 - \beta_2) b_2 + \cdots + (\alpha_n - \beta_n) b_n = \vec{0}.$$

By the linear independence of B, $\alpha_i - \beta_i = 0$ for each i, so $\alpha_i = \beta_i$ for each i. This gives the result.

5. **Bonus** (2 points) : Note that in view of the result you proved in 4, for any finite basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ of a vector space V, we obtain a well defined function $L_{\mathcal{B}} : V \to \mathbb{R}^n$, defined by

$$L_{\mathcal{B}}(\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T.$$

New terminology: We call an invertible linear map an **isomorphism**.

Prove that $L_{\mathcal{B}}$ is an isomorphism.

Answer: To show that L_β is linear for any $v, v' \in V$, write

$$v = \alpha_1 b_1 + \cdots \alpha_n b_n,$$
$$v' = \alpha_1' b_1 + \cdots \alpha_n' b_n.$$

$$\begin{aligned} L_{\mathcal{B}}(v+v') &= L_{\mathcal{B}}((\alpha_1+\alpha_1')b_1 + \cdots + (\alpha_n+\alpha_n')b_n) \\ &= ((\alpha_1+\alpha_1'), \ldots, (\alpha_n+\alpha_n')) \\ &= (\alpha_1, \ldots, \alpha_n') + (\alpha_1', \ldots, \alpha_n') \\ &= L_{\mathcal{B}}(v) + L_{\mathcal{B}}(v'). \end{aligned}$$

A similar argument shows that $L_{\mathcal{B}}(cv) = cL_{\mathcal{B}}(v)$ for all scalars c. Hence $L_{\mathcal{B}}$ is linear.

Further, $L_{\mathcal{B}}$ is invertible: $L_{\mathcal{B}}^{-1} : \mathbb{R}^n \to V$ is given by

$$L_{\mathcal{B}}^{-1}(\alpha_1, \ldots, \alpha_n) = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n.$$

It's clear that $L_{\mathcal{B}} \circ L_{\mathcal{B}}^{-1}(v) = \mathrm{Id}_{\mathbb{R}^n}$ and $L_{\mathcal{B}}^{-1} \circ L_{\mathcal{B}} = \mathrm{Id}_{\mathrm{V}}$, so that these functions are in fact inverses.