Math 4242 Sec 40 Quiz 6

Name: ____________________________

1. (3 points) For $A = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 3 \\ -1 & 1 & 1 & -5 \\ 0 & -1 & 0 & 2 \end{pmatrix}$, find a set of vectors S such that Span(S) $= \mathcal{N}$(A), where $\mathcal{N}(A)$ denotes the null space of A.

2. (2 points) Is

$$v = \begin{pmatrix} 3 \\ 3 \\ -5 \\ 2 \end{pmatrix} \in \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}?$$

Justify your answer. [Hint: You should be able to leverage some of your work on problem 1 to solve this problem.]

3. (2 points) Prove that in any vector space V, the cancellation law holds: For $a, b, c \in V$, if $a + b = a + c$, then $b = c$. Show all steps, and be clear about how you are using the vector space axioms.

Answer: One of the vector space axioms tells us that for any $v \in V$, there exists some element $-v \in V$ with $v + (-v) = 0$; for any $v_1, v_2 \in V$, $v_1 - v_2$ is defined as $v_1 + (-v_2)$. We have
$a + b = a + c \implies (a + b) - a = (a + c) - a$. By commutativity of addition, then, $(a - a) + b = (a - a) + c \implies \vec{0} + b = \vec{0} + c$. But for any $v \in V$, $\vec{0} + v = v$, so we have that $b = c$ as desired.

4. (3 points) Prove that for any field F and vector space V, $\alpha\vec{0} = \vec{0}$ for all $\alpha \in F$. Show all steps, and be clear about how you are using the vector space axioms. [Hints: You may want to use the cancellation law from problem 2. You may also find it helpful to use the fact that $\vec{0} + \vec{0} = \vec{0}$.]

Answer: The answer is given as a part of the solution to the first question in the supplement to homework 6.

5. (**Bonus**, 2 points) For $T : V \to W$ a linear map between vector spaces, prove that $\ker(T) = \{\vec{0}\}$ if and only if T is 1-1.

Answer: First, to be clear, 1-1 here means the same thing as "injective." If T is 1-1, then for every $w \in W$, there is at most one $v \in V$ with $T(v) = w$. Taking $w = \vec{0}$, we have that $\ker(T)$ contains at most one element. Thus it suffices to check that for any linear map T, $T(\vec{0}) = \vec{0}$. (Note that I am using $\vec{0}$ to denote both the zero vector in V and the zero vector in W.) $T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0})$, so by the cancellation law, $T(\vec{0}) = \vec{0}$. Thus $\ker(T) = \{\vec{0}\}$.

Conversely, suppose $\ker(T) = \{\vec{0}\}$. We need to show that for any v_1, v_2 with $T(v_1) = T(v_2)$, $v_1 = v_2$. $\vec{0} = T(v_1) - T(v_2) = T(v_1 - v_2)$, so $v_1 - v_2 \in \ker(T)$. But since $\ker(T) = \{\vec{0}\}$, we have $v_1 - v_2 = \vec{0}$. Thus $v_1 = v_2$.