Math 4242 Sec 40 Quiz 5 Solutions

1. (6 points) For $A = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 1 & 2 & 1 & 0 \\ 2 & 6 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$, find a unit lower triangular matrix L, an upper triangular matrix U, and a permutation matrix P such that $PA = LU$. [A permutation matrix is a matrix whose rows are a permutation of the rows of the identity.]

Using this, solve $A\mathbf{x} = \vec{b}$ for each of the following vectors b:

$$\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Answer: To answer this, we follow the method of Examples 3.10.4 and 3.10.5 of Meyer, which was discussed in class (though perhaps a bit breezily). We perform Gaussian elimination on A. We also keep a vector O which is initially equal to $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, and every time we perform a row swap on our working matrix, we swap the corresponding entries of this vector. At the end of the computation, O encodes P: P_{i*} will be the O_i^{th} row of the 4×4 identity matrix I_4. [It is also possible to construct P directly, by performing successive row swaps on the identity matrix, but Meyer's way is more efficient/convenient.]

To compute L, every time we subtract α [row j] from [row i], we set the $(i,j)^{th}$ entry of the working matrix to α. (I will put such entries in bold to distinguish them from the rest of the entries.) When we perform an operation of type I, these "extra" entries also move. (However, they do not participate in a row operation of type III.) L is the unit lower triangular matrix whose part below the diagonal is the same as that of the matrix obtained when we finish the computation, and U is the upper triangular part of that matrix.

$$\left(\begin{array}{cccc|c} 1 & 4 & 2 & 3 & 1 \\ 1 & 2 & 1 & 0 & 2 \\ 2 & 6 & 3 & 1 & 3 \\ 0 & 0 & 1 & 4 & 4 \end{array}\right) \begin{array}{c} r_2 - 1r_1 \\ r_3 - 2r_1 \end{array} \left(\begin{array}{cccc|c} 1 & 4 & 2 & 3 & 1 \\ \mathbf{1} & -2 & -1 & -3 & 2 \\ \mathbf{2} & -2 & -1 & -5 & 3 \\ \mathbf{0} & 0 & 1 & 4 & 4 \end{array}\right) r_3 - (-1)r_2$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 2 & 3 & 1 \\ \mathbf{1} & -2 & -1 & -3 & 2 \\ \mathbf{2} & \mathbf{-1} & 0 & -2 & 3 \\ \mathbf{0} & \mathbf{0} & 1 & 4 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 4 & 2 & 3 & 1 \\ \mathbf{1} & -2 & -1 & -3 & 2 \\ \mathbf{0} & \mathbf{0} & 1 & 4 & 4 \\ \mathbf{2} & \mathbf{-1} & \mathbf{0} & -2 & 3 \end{array}\right).$$

We thus have

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 4 & 2 & 3 \\ 0 & -2 & -1 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now let's solve the equation $A\mathbf{x} = \vec{b}$, for $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. [I will not write up the solution to the second linear solve you were asked to do, but after seeing the first you should have no trouble.] We have $PA\mathbf{x} = LU\mathbf{x} = Pb$.

We solve $L\mathbf{y} = \vec{P}b$ and then $U\mathbf{x} = \mathbf{y}$. Forward-solving gives $\mathbf{y} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Back-solving then gives $\mathbf{x} = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$.

2. (1 point) What is the inverse of the following product:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -12 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Answer: These sorts of things were discussed at length in class. The answer is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 12 & 2 & -4 & 1 \end{pmatrix}.$$

3. (2 points) Recall from class that $\mathbb{R}^3$ is a vector space over the field $\mathbb{R}$. Which of the following subsets of $\mathbb{R}^3$ are subspaces?

(a) the 1-element set $\{(0, 0, 0)\}$,
(b) the 1-element set $\{(1, 1, 1)\}$,
(c) the z-axis (that is, the set $\{(0, 0, r) \mid r \in \mathbb{R}\}$),
(d) The set of all points in $\mathbb{R}^3$ for which each coordinate is an integer.

answer: (a) and (c).

4. (1 point) Let $\mathbb{Z}$ denote the set of integers. For scalars in $\mathbb{R}$, suppose I define a scalar multiplication operation $\#$ on $\mathbb{Z}$ by

$$r \# z = \begin{cases} rz & \text{if } r \in \mathbb{Z} \\ z & \text{if } r \notin \mathbb{Z} \end{cases}$$

for all $r \in \mathbb{R}$, $z \in \mathbb{Z}$. For example, $3 \# 2 = 6$ and $\sqrt{2} \# 2 = 2$.

For scalar multiplication given by $\#$, and the usual definition of addition in $\mathbb{Z}$, is $\mathbb{Z}$ a vector space over $\mathbb{R}$? Explain your answer.

Answer: No. The associativity property of scalar multiplication does not hold. For example, $2 \# (\frac{1}{2} \# 7) = 14$ but $(2 \times \frac{1}{2}) \# 7 = 1 \# 7 = 7$.