Math 4242 Sec 40 Quiz 2 Partial Solutions
(Problems 4+5)

1. (1.5 points) Which of the following numbers can be represented EXACTLY in base-10 floating point arithmetic with 3 digits of precision? (Identify all of them.)

(a) 2.201
(b) .000304
(c) 100.04

2. (3 points) Use 3-digit, base-10 floating point arithmetic and Gaussian elimination with partial pivoting to obtain an approximate solution the system

$$\begin{aligned} -10^{-5}x + y &= 1 \\ x + 2y &= 2. \end{aligned}$$

3. (2.5 points) Find the general solution to the system

$$\begin{aligned} -x_1 + 2x_2 + x_3 - x_4 + x_5 &= 0 \\ - x_2 - x_3 + x_4 + x_5 &= 0 \\ 2x_1 - x_2 + x_3 \qquad - x_5 &= 0 \end{aligned}$$

4. (1 point) Building on your solution to problem 3, give the general solution to the system

$$\begin{aligned}
-x_1 + 2x_2 + x_3 - x_4 + x_5 &= -1 \\
- x_2 - x_3 + x_4 + x_5 &= 1 \\
2x_1 - x_2 + x_3 \qquad - x_5 &= 0
\end{aligned}$$

Solution. It's easy to see by inspection that (0,0,0,1,0) is a solution to this system. Thus, the general solution to this system is given by $(0,0,0,1,0) + G_h$ where, G_h is the general solution to the corresponding homogeneous system, which you already computed in problem 3.

5. (2 points) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For which vectors $\vec{b} \in \mathbb{R}^3$ is the set of solutions to the system $A\vec{x} = \vec{b}$ closed under addition?

(The set of solutions to $A\vec{x} = \vec{b}$ is said to be *closed under addition* if for any pair $\vec{x}_1$, $\vec{x}_2$ of solutions to $A\vec{x} = \vec{b}$, $\vec{x}_1 + \vec{x}_2$ is also a solution to $A\vec{x} = \vec{b}$.) HINT: If the solution set to $A\vec{x} = \vec{b}$ is empty, then it is closed under addition. When is a non-empty set of solutions to $A\vec{x} = \vec{b}$ closed under addition?

Solution. Let $S(A\vec{x} = \vec{b})$ denote the solution set to $A\vec{x} = \vec{b}$. There are two cases we need to consider: First, it may be that $S(A\vec{x} = \vec{b})$ is closed under addition because it is empty. Second, we may have that $S(A\vec{x} = \vec{b})$ is closed under addition and non-empty. We need to find the possible values of $\vec{b}$ for both cases.

First, let's find all the values of $\vec{b}$ such that $S(A\vec{x} = \vec{b})$ is empty. If $\vec{b} = (b_1, b_2, b_3)$, with $b_3 \neq 0$, then $S(A\vec{x} = \vec{b})$ is empty because the third equation in the system is $0 = b_3$, which has no solution.

On the other hand, if $b_3 = 0$, then performing a backsolve gives that the general solution to $A\vec{x} = \vec{b}$ is

$$A = \begin{pmatrix} -b_1 - b_2 \\ b_2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

so if $b_3 = 0$, $S(A\vec{x} = \vec{b})$ is non-empty. This shows that $S(A\vec{x} = \vec{b})$ is empty if and only if $b_3 = 0$. [Note: by the same reasoning, any time a system in row echelon form **does not** have any equation of the form $0 = c_1$ for $c_1 \in \mathbb{R}$ $c_1 \neq 0$, the system has at least one solution.]

Now suppose that $S(A\vec{x} = \vec{b})$ is nonempty, and let both $\vec{s}$ and $\vec{t}$ be solutions to $A\vec{x} = \vec{b}$. We have $A(\vec{s} + \vec{t}) = A\vec{s} + A\vec{t} = \vec{b} + \vec{b} = \vec{2}b$. If $S(A\vec{x} = \vec{b})$ is also closed under addition, then we also have that $A(\vec{s} + \vec{t}) = \vec{b}$, implying that $\vec{b} = 2\vec{b}$, i.e., $\vec{b} = 0$. Thus, if $S(A\vec{x} = \vec{b})$ is non-empty and closed under addition, then $\vec{b} = 0$.

Conversely, if $\vec{b} = 0$ then $A(\vec{s} + \vec{t}) = A\vec{s} + A\vec{t} = \vec{0} + \vec{0} = \vec{0}$, so $S(A\vec{x} = \vec{b})$ is closed under addition. In summary, if $S(A\vec{x} = \vec{b})$ is non-empty, then $S(A\vec{x} = \vec{b})$ is closed under addition if and only if $\vec{b} = 0$.

Putting together our analyses of cases $S(A\vec{x} = \vec{b})$ empty and $S(A\vec{x} = \vec{b})$ nonempty, we have that $S(A\vec{x} = \vec{b})$ is closed under addition if and only if $\vec{b} = 0$ or $b_3 \neq 0$.