Math 4242 Sec 40 Partial Solutions/Remarks: Quiz 1

1. (3 points) Solve the following linear system using Gaussian elimination (not Gauss-Jordan elimination), together with a backsolve:

$$\begin{aligned} - \quad y + z &= 1 \\ -x + 2y - z &= 1 \\ 2x - \quad y \qquad &= 1 \end{aligned}$$

2. (2 points) Building on your computation in problem 1, use Gauss-Jordan elimination to find the reduced echelon form of the matrix

$$\begin{pmatrix} 0 & -1 & 1 & 1 \\ -1 & 2 & -1 & 1 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

Remark: Remember that you can read off the solution to a linear system directly from its reduced echelon form. So a good way to check for a mistake is to compare the solution obtained from the reduced echelon form with the solution you got from problem 1 via backsolve. If the two solutions are different, you made a mistake somewhere.

3. (1 point) Building on your work in problem 2, write down the reduced echelon form of the matrix

$$\begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 2 & -1 & 1 \end{pmatrix}$$

Answer (with explanation): Recall from class and Section 2.2 of the text that if two matrices A and B are related by row operations, then $E_A = E_B$. Now note that the matrix in this problem is the same as the one from problem 2, except that the bottom row has moved to the top. That means that we can transform one matrix to the other via two elementary row operations of type 1. Therefore, the two matrices have the same reduced row echelon form. So the answer to problem 3 is the same as the answer to problem 2.

4. (2 points) Write down all of the *binary* 2×2 matrices in row echelon form; we say a matrix is binary if each of its entries is either 1 or 0.

Answer: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$

Remark: The informal description of row echelon form in terms of the "staircase" separating the zero elements from the non-zero elements provides good visual intuition for what row-echelon form typically looks like, but when there is doubt about whether a matrix is in row echelon form, one ought to appeal to the **formal definition** of row echelon form.

For example, the formal definition makes clear that the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in row echelon form, but this is hard to see from the informal description.

5. (2 points) Using the definition of row echelon form (as given in class or on page 44 of Meyer), show that a matrix in row echelon form has at most one pivot in each column.

(Reminder: For a matrix E in row echelon form, we say E_{ij} is a *pivot* if E_{ij} is the first non-zero entry in its row; that is, E_{ij} is a pivot if

(1) $E_{ij} \neq 0$,
(2) $E_{ij'} = 0$ for $j' < j$.)

Answer: Suppose that E_{ij} is a topmost pivot, i.e., a pivot such that there is no $i' < i$ with $E_{i'j}$ also a pivot. The definition of row echelon form tells us that $E_{i'j} = 0$ for all $i' > i$. Since a pivot is non-zero by definition, there can be no $i' > i$ with $E_{i'j}$ a pivot. Thus E_{ij} is the only pivot in its column.

Since a topmost pivot is always the only pivot in its column, no column can contain more than one pivot.