Math 4242 Sec 40 Homework 6 Supplement

1. Let V be an arbitrary vector space over a field F. For $\alpha \in F$ and $\vec{v} \in V$, show that $\alpha F = \vec{0}$ if and only if $\alpha = 0$ or $\vec{v} = \vec{0}$. Be explicit about which vector space axioms you are using.

Answer: Recall the cancellation property for vector spaces proven in class: For any $\vec{a}, \vec{b}, \vec{c} \in V$, if $\vec{a} + \vec{b} = \vec{a} + \vec{c}$ then $\vec{b} = \vec{c}$.

Using this, we first show that $0\vec{v} = \vec{0}$ for all $\vec{v} \in V$. We have

$$0\vec{v} = (0+0)\vec{v} = 0\vec{v} + 0\vec{v}.$$

Note also that $0\vec{v} = 0\vec{v} + \vec{0}$ by (A4). Thus we have $0\vec{v} + \vec{0} = 0\vec{v} + 0\vec{v}$. Now we apply the cancellation law to this equation, with $\vec{a} = 0\vec{v}$, $\vec{b} = \vec{0}$, $c = 0\vec{v}$, which gives $\vec{0} = 0\vec{v}$ as desired.

Next we show that $\alpha\vec{0} = \vec{0}$ for all $\vec{\alpha} \in 0$. The proof is similar to the above:

$$\alpha\vec{0} + \vec{0} = \alpha\vec{0} = \alpha(\vec{0} + \vec{0}) = \alpha\vec{0} + \alpha\vec{0}$$

by (A4) and (M4). Now apply the cancellation law as above.

We have shown that if $\alpha = 0$ or $\vec{v} = \vec{0}$, then $\alpha\vec{v} = \vec{0}$. It remains to show the converse. Suppose that $\alpha\vec{v} = \vec{0}$, and also that $\alpha \neq 0$. We check that then $\vec{v} = \vec{0}$. Multiplying both sides of the equation $\alpha\vec{v} = \vec{0}$ on the left by $\frac{1}{\alpha}$ gives $\frac{1}{\alpha}(\alpha\vec{v}) = \frac{1}{\alpha}\vec{0} = \vec{0}$, where the last equality follows from what we showed above. But note that by (M2) and (M5),

$$\frac{1}{\alpha}(\alpha\vec{v}) = (\frac{1}{\alpha}\alpha)\vec{v} = 1\vec{v} = \vec{v}.$$

Thus we have $\vec{v} = \vec{0}$.

Now assume that $\alpha\vec{v} = \vec{0}$ and $\vec{v} \neq 0$. We just showed that if $\alpha \neq 0$, then $\vec{v} = \vec{0}$, so we must have that $\alpha = 0$. We have now shown that if $\alpha F = \vec{0}$ then either $\alpha = 0$ or $\vec{v} = \vec{0}$. This completes the proof. □

2.Let $v_1 \in \mathbb{R}^3$ be a non-zero vector and suppose we have $v_2 \in \mathbb{R}^3$ such that $v_2 \notin \text{Span}\{v_1\}$. Let $w = v_1 \times v_2$ (the cross product of v_1 and v_2). Show that $\text{Span}\{v_1, v_2\} = P$, where

$$P = \{v \in \mathbb{R}^3 \mid w \cdot v = 0\},$$

the plane perpendicular to w and passing through $\vec{0}$. [Hint: Show $\text{Span}\{v_1, v_2\} \subseteq P$ and $P \subseteq \text{Span}\{v_1, v_2\}$.]

Answer: For any $a, b \in \mathbb{R}$, $w \cdot (av_1 + bv_2) = aw \cdot v_1 + bw \cdot v_2 = 0 + 0 = 0$, because a cross product of two vectors is perpendicular to both vectors. Thus any linear combination of v_1, v_2 is in P. Hence $\text{Span}\{v_1, v_2\} \in P$.

With a little more vector space theory, it would be very quick to check that, in fact, $\text{Span}\{v_1, v_2\} = P$. An informal argument would be: $\text{Span}\{v_1, v_2\}$ is a subspace of P which strictly contains a line, so it is a plane. Therefore it must be equal to the whole plane P. For now, this is good enough for our purposes.

Here is a more rigorous (but laborious) proof in the "Meyer style," using the machinery we have. First, we observe that $\text{Span}\{v_1, v_2, w\} = \mathbb{R}^3$. To show this, we argue by contradiction: If $\text{Span}\{v_1, v_2, w\} \neq \mathbb{R}^3$, then there is some vector $z \in \mathbb{R}^3$ that is not a linear combination of $\{v_1, v_2, w\}$. Let $A = (v_1|v_2|w|z)$. v_2 is not a multiple of v_1 by assumption. Also since w is perpendicular to both v_1 and v_2, w is not a linear combination of v_1 and v_2. Hence, no column of A is a linear combination of previous columns. Thus it must be that every column in A is basic. But since A has only 3 rows, this is impossible. Hence $\text{Span}\{v_1, v_2, w\} = \mathbb{R}^3$.

Thus for any $v \in P$ we may write $P = av_1 + bv_2 + cw$ for $a, b, c \in \mathbb{R}$. But then

$$0 = w \cdot v = aw \cdot v_1 + bw \cdot v_2 + cw \cdot w = cw \cdot w.$$

$w \cdot w \neq 0$, so we must have $c = 0$. Thus $v \in \text{Span}\{v_1, v_2\}$. This shows that $P \subset \text{Span}\{v_1, v_2\}$. We conclude that $P = \text{Span}\{v_1, v_2\}$ as claimed.