Math 4242 Sec 40 Homework 4 Supplement

1. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear function given by

$$F(x, y, z) = \begin{pmatrix} y + z \\ x + z \\ x + y \end{pmatrix}$$

and let $G : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear function given by

$$G(x, y, z) = \begin{pmatrix} x + z \\ y + z \end{pmatrix}.$$

Find a 2×3 matrix A such that for any $\vec{v} \in \mathbb{R}^3$, $G \circ F(\vec{v}) = A\vec{v}$.

Answer:

$$G \circ F(x, y, z) = G(y+z, x+z, x+y) = \begin{pmatrix} x + 2y + z \\ 2x + y + z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

So

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

2. In the problem above, what is the relationship between A and the matrices $[F]$ and $[G]$?

Answer: Note that A is the matrix such that $T_A = G \circ F$, for $T_A : \mathbb{R}^3 \to \mathbb{R}^2$ as defined in class. So by results from class, $A = [T_A] = [G \circ F] = [G][F]$.

3. Write down the 3×3 matrix E such that for any matrix A with 3 rows, EA is the matrix obtained from A by first adding $3A_{2*}$ to A_{3*} and then switching the rows A_{2*} and A_{1*}.

Answer: Let E_1 be the matrix such that $E_1 A$ is the matrix obtained from A by adding $3A_{2*}$ to A_{3*}, for any matrix A w/ 3 rows.

Let E_2 be the matrix such that $E_2 A$ is the matrix obtained from A by switching the rows A_{2*} and A_{1*}, for any matrix A w/ 3 rows.

Then $E = E_2 E_1$. Recall from Friday that to find the elementary matrix associated to an row elementary operation, we apply that row operation to the identity matrix. Thus,

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$

4. We define elementary column operations of Types I—III on a matrix in a way exactly analogous to the way we defined elementary row operations. That is,

(1) an elementary column operation of type I switches two columns,
(2) an elementary row operation of type II multiplies a column by a non-zero scalar,
(3) an elementary column operation of type III adds a multiple of one column to another column.

Explain why for any matrix A, we can perform any elementary column operation on A by multiplying A on the **right** by an elementary matrix of the form considered in class on Friday.

HINTS: Since transposes convert rows to columns and columns to rows, performing an elementary column operation on A is the same as performing an elementary row operation on A^T and then taking the transpose of the resulting matrix. Use the facts that for any matrices A, B whose product AB is defined, we have $(AB)^T = B^T A^T$ and $((AB)^T)^T) = AB$. You will also want to use the fact that the transpose of each of the elementary matrices $P_n^{i,j}$, $Q_n^{i,\alpha}$, $R_n^{i,j,\alpha}$ introduced in class is an elementary matrix of the same type.

Answer: We'll show that to perform an elementary column operation on A, we can multiply A on the right by E^T, where E is the matrix representation of the corresponding row operation. For example, if E is the matrix such that multiplying A on the left by E adds $3A_{2*}$ to A_{5*}, then multiplying A on the right by E^T has the effect of adding $3A_{*2}$ to A_{*5}.

As noted in the hint, to perform an elementary column operation on A, we can perform the corresponding elementary row operation on A^T and then take the transpose of the resulting matrix. Let E be the elementary matrix such that multiplying A^T on the left by E performs the corresponding row operation. Then $(EA^T)^T$ is matrix obtained after performing our elementary column operation on A. But $(EA^T)^T = (A^T)^T E^T = AE^T$. Thus multiplying A on the right by E^T performs the column operation on A.

As mentioned in the hint, E^T is an elementary matrix of the same type as E. (That is for $j = 1, 2, 3$, E^T is an elementary matrix of Type j iff E is an elementary matrix of Type j; check this.) So we are done.

5. Find a 4×4 matrix E such that that for any matrix A with 4 columns, AE is the matrix obtained from A by adding the second column of A to the fourth column.

Answer: For any matrix B with 4 rows, $R_4^{4,2,1}B$ is the matrix obtained from B by adding the second row of A to the fourth row, where as defined in class,

$$R_4^{2,4,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

By the answer to question 4, then, we have that

$$E = (R_4^{4,2,1})^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R_4^{2,4,1}.$$

It's also easy to see that this is the right answer directly from the column-wise definition of matrix multiplication. (You should check this.)