Overview of Stochastic Approaches to Social Network Analysis

Wasserman and Faust, Chapter 13-16.


Unit of analysis & Five Distinct levels of analysis

• Unit of analysis in network analysis:
  not the individual, but an entity consisting of a collection of individuals and the linkages among them

  o Individual actor level of analysis
  o Dyads level of analysis (two actors and their ties)
  o Triad level of analysis (three actors and their ties)
  o Subgroup level of analysis
  o Global level of analysis

# Network properties at different levels of analysis

<table>
<thead>
<tr>
<th>Level of analysis</th>
<th>Network properties</th>
</tr>
</thead>
</table>
| Individual actor level  | • Degree, Indegree, and Outdegree  
                           • Betweenness  
                           • Closeness  
                           • Centrality and prestige  
                           • Roles (isolates, liaisons, bridges)  
                           • Structural holes (Burt, 1992) |
| Local level             |                                                                                   |
| Dyads level             | • Distance and Geodesics  
                           • Structural equivalence                                                       |
| Triad level             | • Transitivity  
                           • Cyclicity                                                                        |
| Subgroup level          | • Component and cliques  
                           • Positions                                                                      |
| Global level            | • Connectedness and diameter  
                           • Density  
                           • Prestige  
                           • Network centralization                                                          |
Ch. 13-14
Dyadic and Triadic Methods

Wasserman and Faust
Dyadic and Triadic Methods

• Level of analysis: “local”
  ○ Look at subgraphs embedded within the graph for the entire network

• Dyads: subgraphs of size 2 consisting of a pair of actors and all ties between them
  ○ Basic unit for the statistical analysis of social networks

• Triads: subgraphs of size 3 consisting of a triple of actors and all ties among them
  ○ Balance theory (Fritz Heider, 1946)

“...the fact that two elements [in a triad] are each connected not only by a straight line—the shortest—but also by a broken line, as it were, is an enrichment form a formal-sociological standpoint.” Simmel (1908)
Reciprocity & Structural balance

• Reciprocity (mutuality)
  o “How strong is the tendency for one actor to “choose” another, if the second actor chooses the first?”

• Structural balance
  o Balance theory (Fritz Heider, 1946)
    ➢ Whether the triad is transitive (structural balance)
    ➢ Whether the triad is balanced (structural transitivity)
To test statistically propositions about a theory...

- One needs a probability distributions.
- Based on a probability distributions,
  - Such models allow the data to show some error, or lack of fit to structural theories, but still support the theories under study.
  - Adoption of this probabilistic approach implies that we can allow a given social network to exhibit a “little bit” of intransitivity, and still be able to conclude that, overall, the network adheres to a theory of transitive triads.
  - We can ask how much intransitivity a network can have before concluding that it is not really a transitive network.

- Deterministic model and Statistical model
Overview of Ch. 13

• *Measurement* for the degree of reciprocity in a network
• Discussion of the *Dyad census*
  o The counts of the different types of dyads that can occur
  o The expected value of the numbers of these different types of dyads (assuming that specific distributions are appropriate)
  o Tests for hypotheses about the number of choices and the number of mutual choices on a specific relation

• *Random directed graph probability distributions*
  o “What distribution do my random variables follow?”
  o These distributions allow a researcher to test hypotheses about various properties of a directed graph under study, such as the number of mutual dyads.
Dyads

• A dyad is an unordered pair of actors and the arcs that exist between the two actors in the pair., $D_{ij} = (X_{ij}, X_{ji})$
  o Given “$g$” actors,
    ➢ $g(g-1)/2$ dyads (unordered pairs of actors); $g(g-1)$ ordered pairs of actors

• Three dyadic isomorphism classes or states
  o $M$: Mutual relationships
  o $A$: Asymmetric dyads (2 types)
  o $N$: Null dyad
Dyads

• The entries in the sociomatrix X will be viewed as random binary variables.
  ➢ The dyads also has bivariate random quantity \((X_{ij}, X_{ji})\).
  ➢ A pair of binary random variables has four states or realizations, depending on the arcs that are present or absent in the dyad, \(D_{ij} = (X_{ij}, X_{ji})\).
  ➢ But, three isomorphism classes for a dyad.

• However, in the Ch.15., we will expand the dichotomous relations to the discrete, valued relations.

\[
\begin{align*}
\bullet & \leftrightarrow \bullet \\
& n_i & n_j \\
D_{ij} &= (1,1) & \text{Mutual dyads} \\
\bullet & \rightarrow & \bullet \\
& n_i & n_j \\
D_{ij} &= (1,0) & \text{Asymmetric dyads} \\
\bullet & \leftarrow & \bullet \\
& n_i & n_j \\
D_{ij} &= (0,1) & \text{Asymmetric dyads} \\
\bullet & & \bullet \\
& n_i & n_j \\
D_{ij} &= (0,0) & \text{Null dyads}
\end{align*}
\]
Dyad Census

\[ <M, A, N> \]

- \[ M = \sum X_{ij} X_{ji} \]
- \[ A = X_{++} - 2M \]
  
  where \( X_{++} = L \), the number of arcs in the digraph

- \[ N = \left[ \frac{g(g-1)}{2} \right] - A - M \]
An index for Mutuality

- Katz and Powell (1955)’s Index ($q_{KP}$)
  - to measure the tendency for actors in a group to reciprocate choices more frequently than would occur simply by chance.
  - The strength of the tendency toward reciprocation of choice
  - $-\infty < q_{KP} \leq 0$
    - 0 = no tendency for choices to reciprocation
    - 1 = maximal tendency
    - negative values = tendencies away from mutual dyads towards asymmetrics and nulls
  - Given fixed choice (investigator suggests a fixed number of actors) and free choice (“List all your best friends”) as a data collection methods
An index for Mutuality

• Fixed choice situation
  - 13.11
    \[ \hat{\rho}_{KP} = \frac{2(g - 1)M - gd^2}{gd(g - 1 - d)}. \]

• Free choice situation
  - 13.14
    \[ \hat{\rho}'_{KP} = \frac{2(g - 1)^2M - L^2 + L_2}{L(g - 1)^2 - L^2 + L_2}. \]
An second index for Mutuality

• Achuthan, Rao, and Rao (1982)’s Index($q_B$)
  o Standardized measure of the level of mutuality in a social network analysis
  o Range of the number of mutual dyads that can arise in a directed graph with specified outdegrees
  o Depending on the numbers of choices made, and its range of possible values is restricted by the out degrees
  o $M_{\text{min}} \leq M \leq M_{\text{max}}$

  o 13.15

\[ \rho_B = \frac{M - M_{\text{min}}}{M_{\text{max}} - M_{\text{min}}} \]
Simple Distributions on Diagraphs

• $G_d(N)$ : a set of all possible labeled and irreflexive directed graphs with “g” nodes

• Uniform distributions
  o Uniform probability function is,
    ➢ $P(X=x) = \frac{1}{2^g(g-1)}$

• Bernoulli distribution
  ➢ $P(X_{ij} = 1) = \begin{cases} P_{ij}, & i \neq j \\ 0, & i = j \end{cases}$
  ,where $0 \leq P_{ij} \leq 1$

“If random digraph follows the Bernoulli distribution and $P_{ij} = \frac{1}{2}$ for all $i \neq j$, then the random digraph is uniformly distributed. If a set of $P_{ij}$ are all equal, but not equal to $1/2$, the distribution is not uniform.”
Validity of the distribution assumption:

• Is the uniform distribution a realistic assumption for X?

• Uniform distribution: X~U.
  - The elements of X are independent
  - Constant probability of 1/2 of being unity
  - L = the count of how many of these Bernoulli random variables are unity

• Under the appropriate uniform distribution,
  - L is a binomial random variable with parameter g(g-1) and P=1/2; that is, L ~ Bin(g(g-1), 1/2).
  - E(L) = (1/2)g(g-1)
  - Var(L) = (1/4)g(g-1)

  \[ z_l = \frac{l - E(L)}{\sqrt{Var(L)}} = \frac{l - g(g - 1)/2}{\sqrt{g(g - 1)/4}} \]
Testing

• Under the assumption that the elements of X are independent with constant probability of 1/2 of being unity
  
  $H_0$: $L \sim \text{Bin}(g(g-1), 1/2)$.  
  
  - If rejecting $H_0$, uniform distribution is not appropriate.  
  - But, we cannot, determine the reason.

• Under the unknown $P$
  
  - For estimating $P$, mean and variance of $L$ from the data  
  - Assuming that $P$ is equal to $P_0$ (unknown parameter)  
  - $X \sim \text{B}$ with a constant probability $P_0$
  
  13.26
  
  $$P(L = l) = \binom{g(g-1)}{l} P_0^l (1 - P_0)^{g(g-1) - l}.$$  
  
  - $E(L) = (P_0)g(g-1)$  
  - $\text{Var}(L) = (P_0 (P_0 - 1))g(g-1)$

  $H_0$: $L \sim \text{Bin}(g(g-1), P_0)$.
  
  - If not rejecting $H_0$, diagraph under study could be distributed as a Bernoulli random diagraph, with known, constant probability of $P_0$ governing the presence/absence of arcs.

  13.28
  
  $$z_l = \frac{l - P_0 g(g-1)}{\sqrt{P_0 (1 - P_0) g(g-1)}}$$
Estimation of unknown $P (P^\wedge)$

- $P^\wedge = l / g(g-1)$
  - where $l$ = number of actually observed arc

- Maximum likelihood estimate,
  - $E(P) = P^\wedge g(g-1) = l$
  - $Var(L) = P^\wedge (1-P^\wedge) g(g-1)$
  - $13.29$
  - $z_l = \sqrt{\frac{P - \hat{P}}{P(1-P)/g(g-1)}}$
  - Confidence interval: $P^\wedge_{\text{lower}} \leq P \leq P^\wedge_{\text{upper}}$
Conditional Uniform Distributions

• Statistical conditioning
  : Restriction of the possible random digraphs that can arise to only those random digraphs with the specific properties conditioned upon, or fixed.
  ➢ Determining characteristics to fix
  ➢ Removing all the digraphs without specific characteristic from $G_d(N)$
  ➢ Constituting revised sample space

• Uniform distribution, conditional on the number of arcs
  ➢ $U \mid L = x_{++}$

• Uniform distribution, conditional on the outdegrees
  ➢ $U \mid \{x_{i+}\}$
Conditional Uniform Distributions

- Uniform distribution, conditional on the number of arcs
  - $U|L = x_{++}$
  - Conditional uniform distribution which gives equal probability to all digraphs with $L$ arcs, and zero probability to all elements of $G_d(N)$ that do not have $x_{++}$ arcs.
  - Sample space (S) including only those digraphs with $L$ arcs.
  - S has their $x_{++}$ arcs in any of $g(g-1)$ “locations.”
  - “How many random digraphs satisfy constraints place on $G_d(N)$ by the conditioning?”

\[
P(X = x) = \begin{cases} 
\frac{1}{(g-1)!}, & \text{if } x_{++} = l \\
0, & \text{otherwise.} 
\end{cases}
\]

- Uniform distribution, conditional on the outdegrees
  - $U|\{x_{i+}\}$
  - Conditional uniform distribution for random directed graphs that conditions on a fixed set of outdegrees. Every directed graph with the specified outdegrees, $X_{i+} = x_{i+}, X_{2+} = x_{2+}, \ldots X_{g+} = x_{g+}$, has equal probability of occurring.
  - Sample space (S) includes all digraphs with exactly the specified outdegrees

\[
P(X = x) = \begin{cases} 
\prod_{i=1}^{g} \left(\frac{1}{(i_{++})!}\right), & \text{if } X_{i+} = x_{i+} \text{ for all } i \\
0, & \text{otherwise.} 
\end{cases}
\]

- $P(X = x) = \left(\frac{(g-1)^{-g}}{d}\right)^g, \text{ if } X_{i+} = d \text{ for all } i,
  0, \text{ otherwise.}$
Statistical Analysis of the Number of Mutuals

- $X \sim U \{x_{i+} = d\}$
- $E(M \mid \{x_{i+} = d\})$
  \[= \left\{ \frac{d^2}{(g-1)^2} \right\} \cdot \left\{ \frac{g(g-1)}{2} \right\} = \frac{gd^2}{2(g-1)}\]
- $Var(M \mid \{x_{i+} = d\}) = \frac{gd^2}{2(g-1)}$
- When all outdegrees are not equal, 13.34

E.g. Suppose that $g=10$, and each actor in the network is instructed to choose four other actors ($d=4$).
  - $E(M \mid \{x_{i+} = 4\}) = \frac{10(4^2)}{(2)(9)} = 8.89$
  - Total number of dyads = $\frac{(10)(9)}{(2)} = 45$
    - We could expect that $\frac{8.89}{45} = 0.1976$ of the dyads will be mutuals.
    - If we see many more than this number, we could conclude that on this relation, the actors reciprocate more than we expect.
Statistical Analysis of the Number of Mutuals: Testing

- Katz, Tagiuri, and Wilson (1956)
  - Conjecture that the distribution is asymptotically Poisson since $E(M | \{x_{i^+} = d\})$ and $\text{Var}(M | \{x_{i^+} = d\})$ are asymptotically equal (when $g > 10d$).
  - But, when $g < 10$ and $d \geq 2$, a normal approximation is better.
- Standard large sample testing approximations to test hypotheses about $M$
  - $G = \frac{(\text{observed } M - \text{hypothesized } M)}{\text{Std} (\sqrt{\text{variance of } M})}$
  - Assuming that this standardized statistic has an approximate normal distribution.
- $H_a$: we have observed too many mutuals to support this null distributions assumption.

- E.g. Suppose that $M=23$, $E(M | \{x_{i^+}\}) = 11.90$, $\text{Var}(M | \{x_{i^+}\}) = 3.784^2$
  - $(23 - 11.90)/3.784 = 2.933$, which has a p-value for a one-tailed test of 0.0017.
  - Rejecting the $H_0$
  - Concluding that the number of mutual dyads observed may be too large, given the null distribution. So, this network has more reciprocity than expected by chance.
  - Suggesting strong tendency about reciprocity
Other conditional Uniform Distributions

- Uniform distribution, conditional on indegrees
- Uniform distribution, conditional on the number of mutual, asymmetric, and null dyads
- Uniform distribution, conditional on outdegrees simultaneously
- Uniform distribution, conditional on outdegrees and number of mutuals
- Uniform distribution, conditional on outdegrees, indegrees, and number of mutuals
Other conditional Uniform Distributions

- Uniform distribution, conditional on indegrees
  - $U | \{X_{+j} = x_{+j} \}$
  - Conditional uniform distribution for random directed graphs that conditions on a fixed set of indegrees. Every directed graph with the specified indegrees, $X_{+1} = x_{+1}, X_{+2} = x_{+2}, \ldots X_{+g} = x_{+g}$, has equal probability of occurring.
  - Sample space $(S)$ includes all digraphs with exactly the specified indegrees
  - $13.35$
    $$P(X = x) = \begin{cases} \prod_{j=1}^{g} \frac{1}{(x_{+j})}, & \text{if } X_{+j} = x_{+j} \text{ for all } j, \\ 0, & \text{otherwise.} \end{cases}$$
  - Fixing the row totals of a sociomatrix
  - Transposing the matrix (so that the $(j,i)$th element of the transposed matrix is the $(i,j)$th element of the original matrix
  - Obtaining the $U | \{X_{+j} \}$ distribution, where conditioning indegrees are the original, fixed row totals.
Other conditional Uniform Distributions

- Uniform distribution, conditional on the number of mutual, asymmetric, and null dyads
  - U|M=A=M distribution
  - U|M=m, A=a, N=n (actual frequencies of the three types of dyads observed for a particular digraph).
  - \( m+a+n = \frac{g(g-1)}{2} \)
  - “How many ways can we take the dyads and divide them so that the first group has \( m \) dyads, the second \( a \), and the third, the remainder of the dyads, will be null?”
  - Thus, \( \binom{2^a}{g(g-1)/2} / [m! \cdot a! \cdot n!] \) different digraphs with \( m \) mutual, \( a \) asymmetric, and \( n \) null dyads
  - E.g. Suppose that 4 nodes (6 dyads), \( m=2, a=2, n=2 \)
  - 13.36

\[
P(X = x) = \begin{cases} 
\frac{[m! \cdot a! \cdot n!]}{(2^a)(\frac{g(g-1)}{2})!}, & \text{if } M=m, A=a, N=n \\
0, & \text{otherwise,}
\end{cases}
\]

where \( m + a + n = (\frac{g}{2}) \).
More Complex Other Conditional Uniform Distributions

- $U|\{X_{i+}\},\{X_{+j}\}$ distribution
  - Combining $U|\{X_{i+}\}$ and $U|\{X_{+j}\}$
  - Simultaneously conditions on both the indegrees and outdegrees of the digraph
  - “A sociomatrix could have arisen from a random digraph distribution with indegrees and outdegrees corresponding to the two sets?”
  - Row totals + column totals = $L$ (total # of 1’s in the sociomatrix)

- $U|M,\{X_{i+}\}$ distribution
  - Conditioning on the # of mutual dyads and the outdegrees
  - Combination of $U|\text{MAN}$ and $U|\{X_{i+}\}$

- $U|M,\{X_{i+}\},\{X_{+j}\}$ distribution
  - Conditioning on choices made, choices received, and the types of dytads
Ch. 14 Triads

• Structural balance and transitivity theories
• Triads (subgraphs of size 3 consisting of a triple of actors and all ties among them)
• Triad census
  o A set of counts of the different kinds of triads that arise in an observed network
• Structural property as to triads
  o Balance, clustering, ranked clusters, and transitivity
    ➢ Efforts to link structural patterns found in triads (microstructure) to macrostructural patterns (e.g. ranked clusters, partial orderings, and so on)
Random models and Substantive Hypotheses

• What statistical models(s) should be used to study and test for non-randomness in social network data?

• Important nodal and dyad properties in digraph:
  ➢ Nodal outdegrees: to control for possible experimental constraints (fixed choice data)
  ➢ Nodal indegrees: to control for differential popularity
  ➢ Dyadic mutuality: to control for tendencies toward reciprocation of choices

• Structural balance and transitivity are deterministic
  o Specific subgroups contained in the graph theoretic representation of a social network should (not) occur.
Triads

• $T_{ijk}$: Triad, or 3-subgraph with $n_i, n_j,$ and $n_k$
  o Actual order and of the actors matters, let $i < j < k$
  o Six possible ordered triples associated with each triad
  o Given $g$ actors,
    ➢ # of triads = $\frac{1}{6}g(g-1)(g-2)$

• How many isomorphism classes exist for triads?
  ➢ Examining triadic realizations
Triads

Six realizations of the single arc triad

(i) $i \rightarrow j; j \not\leftrightarrow i; i \not\leftrightarrow k; k \not\leftrightarrow i; j \not\leftrightarrow k; k \not\leftrightarrow j$.

(ii) $i \not\leftrightarrow j; j \rightarrow i; i \not\leftrightarrow k; k \not\leftrightarrow i; j \not\leftrightarrow k; k \not\leftrightarrow j$.

(iii) $i \not\leftrightarrow j; j \not\leftrightarrow i; i \rightarrow k; k \not\leftrightarrow i; j \not\leftrightarrow k; k \not\leftrightarrow j$.

(iv) $i \not\leftrightarrow j; j \not\leftrightarrow i; i \not\leftrightarrow k; k \rightarrow i; j \not\leftrightarrow k; k \not\leftrightarrow j$.

(v) $i \not\leftrightarrow j; j \not\leftrightarrow i; i \not\leftrightarrow k; k \not\leftrightarrow i; j \rightarrow k; k \not\leftrightarrow j$.

(vi) $i \not\leftrightarrow j; j \not\leftrightarrow i; i \not\leftrightarrow k; k \not\leftrightarrow i; j \not\leftrightarrow k; k \rightarrow j$. 
Examining triadic realizations

• The six realizations of the single arc triad

However, after erasing all the labels on these six triads?
They are all isomorphic!
The 16 Triad Isomorphism Classes

Seven types: Number of choices made

(0)  (1)  (2)  (3)  (4)  (5)  (6)

003  012  102  111D  201  210  300

M-A-N labeling:
000 = # of Mutual, Asymmetric, Null
U = For Up
D = For Down
T = For Transitive
C = For Cyclic

SOURCE: after James Moody’s slides
**Triad census (T)**

- $T_u =$ the number of triads that belong to isomorphism class $u$
- $T = (T_{003}, T_{012}, \ldots, T_{300})' = (003, 012, 102, 021D, 021U, 021C, 111D, 111U, 030T, 030C, 201, 120D, 120U, 120C, 210, 300)$
Example

- \( (1/6)(6)(5)(4) = 20 \) triads
- \( 003 = (T_{145}, T_{245}, T_{456}) \)
- \( 012 = (T_{135}, T_{146}, T_{156}, T_{235}, T_{356}) \)
- \( 102 = (T_{124}, T_{125}, T_{246}, T_{256}, T_{345}) \)
- \( 111U = (T_{234}, T_{346}) \)
- \( 111D = (T_{134}) \)
- \( 030T = (T_{136}) \)
- \( 120D = (T_{236}) \)
- \( 120C = (T_{123}) \)
- \( 210 = (T_{126}) \)

\[ T = (3, 5, 5, 0, 0, 0, 1, 2, 1, 0, 0, 1, 0, 1, 1, 0) \]
Triads

• Triads are clearly not independent of each other.
• Suppose that e.g., \( T=(3,5,5,0,0,0,1,2,1,0,0,1,0,1,1,0) \)
• Triad census tell us (linear combinations of triad census)
  \[ \sum_{u} l_{u} T_{u}, \]
  where \( l_{u} = \) the coefficients of the linear combination
  \( u = \)ranges over the sixteen triad types \( (1 \leq u \leq 16) \)
  The counts in the dyad census \[ 14.2 \]
  The number of nodes in the digraph \( (g) \)
  Counts of the number of arcs \( (L) [14.3] \)
  \( C_{u} = \) the number of arc in the \( u \)th triad type.
  Mean and variance of the indegrees and outdegrees
  \( M, A, N \) 
  \[ 14.9 \] \[ 14.10 \] \[ 14.11 \]
Weighting vectors for statistics and hypothesis concerning the triad census [Table 14.2]

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<th>Triad type</th>
<th>$b_{in,u}$</th>
<th>$b_{out,u}$</th>
<th>$m_u$</th>
<th>$a_u$</th>
<th>$n_u$</th>
<th>$c_u$</th>
<th>Trans.</th>
<th>Intrans.</th>
<th>Close friends disagreeing</th>
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</tr>
</tbody>
</table>
Distribution of a Triad Census

- **T** has sixteen components, and hence has a covariance matrix which has dimensions $16 \times 16$ and a multivariate distribution which is 16-dimentional.
  - The distribution of **T** can be well-approximated by the multivariate normal distribution.
  - Multivariate statistical inference relies on assumption of approximate multivariate normality.

- Mean and Variance of a k-subgraph Census
  - **K**, belonging to isomorphsim class **u** (**L**, belonging to class **v**)
  - **K** and **L** = distinct two subgraphs of k-subgraphs given g nodes
  - Assuming that the digraph in question in random, so that we will need a notation for the various probabilities that arise with our k-subgraph size.
  - Defining probability that any one of the k-subgraph(**K**) falls into one of the isomorphism classes(**u**),
    - $P_K(u) = P(\text{K is in class } u)$
  - The probability that any pair of the k-subgraphs fall into two particular classes, $P_{K,L}(u,v) = P(\text{K is in class } u \text{ and } L \text{ is in class } v)$
Distribution of a Triad Census

- Expected frequencies of the counts in the \( k \)-subgraph census depend on averages of the \( \{P_K(u)\} \)
  \( \circ (14.14) \)
  \[
  \bar{p}(u) = \frac{1}{\binom{g}{k}} \sum_K p_K(u).
  \]

- Theorem 1. Using the notation given above and assuming that a random digraph is generated by some stochastic mechanism, then the expected number of \( k \)-subgraphs in class \( u \), which we will call \( H_u \)
  \( \circ (14.15) \)
  \[
  E(H_u) = \binom{g}{k} \bar{p}(u).
  \]

- \( \circ (14.16) \)
  \[
  \bar{p}_j(u,v) = \frac{1}{\binom{g}{k} \binom{g-k}{k-j} \binom{k}{j}} \sum p_{k,L}(u,v),
  \]
Distribution of a Triad Census

Theorem 2. Using the notation given above and assuming that a random digraph is generated by some stochastic mechanism, then the variance of the number of $k$-subgraphs in class $u$, is

$$\text{Var}(H_u) = \binom{g}{k} \left\{ \bar{p}(u)(1 - \bar{p}(u)) \right. \\
+ \sum_{j=0}^{k-1} \binom{g-k}{k-j} \binom{k}{j} \left[ \bar{p}_j(u,u) - (\bar{p}(u))^2 \right] \right\}.$$  \hspace{1cm} \textcircled{14.17}

$$\text{Cov}(H_u, H_v) = \binom{g}{k} \left\{ -\bar{p}(u)\bar{p}(v) \right. \\
+ \sum_{j=0}^{k-1} \binom{g-k}{k-j} \binom{k}{j} \left[ \bar{p}_j(u,u) - (\bar{p}(u))^2 \right] \right\}. \hspace{1cm} \textcircled{14.18}$$
Mean and Variance of a Triad Census

\[ E(T_u) = \binom{g}{3} \bar{p}(u) \]  \hfill (14.19)

\[ \text{Var}(T_u) = \binom{g}{3} \bar{p}(u)(1 - \bar{p}(u)) \]
\[ + \binom{g}{3} \sum_{j=0}^{2} \left( \binom{g-3}{3-j} \binom{3}{j} \right) [\bar{p}_j(u, u) - (\bar{p}(u))^2] \]  \hfill (14.20)

\[ \text{Cov}(T_u, T_v) = \binom{g}{3} \left\{ - \bar{p}(u)\bar{p}(v) \right\} \]
\[ + \sum_{j=0}^{2} \left( \binom{g-3}{3-j} \binom{3}{j} \right) [\bar{p}_j(u, u) - (\bar{p}(u))^2] \right\}. \]  \hfill (14.21)

Tu, is one of the sixteen counts of the triad census, and replace Hu in Theorems 14.1 and 14.2.
Mean and Variance of Linear Combination of a Triad Census

• 14.22

\[ E\left(\sum_{u} l_u T_u\right) = \sum_{u} l_u E(T_u) = l' \mu_T \]

• 14.23

\[ \text{Var}\left(\sum_{u} l_u T_u\right) = l' \Sigma_T l. \]
Testing Structural Hypotheses

- Structural Hypotheses
- How these hypotheses can be “operationalized” in terms of triads?
- Configurations: a subset of the nodes and some of the arcs that may be contained in a triad.
  - Translating substantive theories into mathematical statements about triads
  - Configuration for similarity/attraction theory
    - First row lists = reading rule (pair of node)
    - Second row list = presence/absence
    - 12 configurations

  “Friends are likely to agree, and unlikely to disagree; close friends are very likely to agree, and very unlikely to disagree (Mazur, 1971)”
From Configuration to Weighting Vectors

• H: a set of prediction about actor and “choice” behavior on the relation under study
  ➢ How frequently the configurations occur
  ➢ By comparing the actual frequencies to the predicted by configuration

• Researcher
  o Determining which configuration are predicted or not by H.
    ➢ E.g., Ha: According to the similarity/atraction hypothesis, the six configurations (111, 1100, 1011/0111, 1000/0100) should occur much more frequently than “chance”, and the other six(1101/1110,1010/0101, 1001/0110), much less.
    ➢ Note: H$_0$: Assuming that Network does not display similarity of attraction, transitivity, and so on.
  o Considering each of the relevant configurations, and finding all the triads should occur and which one should not.
  o Applying weighting vector, we could calculate the difference between actual frequency and expected frequency
  o Then, using covariance matrix for the triad census, we could judge statistical significance of the difference.
Configuration types for Mazur’s proposition

Seven Weighting Vectors

<table>
<thead>
<tr>
<th>Triad type</th>
<th>1111</th>
<th>1100</th>
<th>1011</th>
<th>0111</th>
<th>1000</th>
<th>0100</th>
<th>1101</th>
<th>1110</th>
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</table>
From Weighting Vectors to Test Statistics

\[ \tau(l) = (l'T - l'[\mu_T]) / \sqrt{l'\Sigma_T l} \]

- The specific configuration associated with the weighting vector compares the observed count of the frequency of this configuration, \( l'T \), to its expected frequency, \( l'[\mu_T] \).
  - \( l \) denotes one of the weighting vectors.
  - \( T \) denotes the triad census vector.
  - \( l'T \) denotes the linear combination of the triad census, using one of the weighting vectors derived from the substantive hypothesis.
  - \( \mu_T \) denotes the mean triad census vector.
- This difference is standardized by the standard error of the configuration frequency to give us an interpretable statistic.
- The above equation assumed to have an approximate normal distribution with mean “0” and variance “1” when the hypothesis under study is true.
**P*** social network models


**$P^*$ family of models**

- $P^*$ family of models
- The formulation presented by Wasserman and Pattison 1996. allows these models to be viewed in a very standard response/explanatory variables setting
- in which the response variable is a logit, or log odds of the probability that a relational tie is present,
- and in which the explanatory variables can be quite general.
- Postulate that the probability of an observed graph is proportional to an exponential function of a linear combination of the network statistics
- the $P^*$ family is easily fit approximately using standard logistic regression modeling software.
**P* family of models**

- The *p* family first arose in spatial modeling. (Wasserman and Pattison, 1996)
- A very general formulation of it,
  - presented by, proceeds by considering a family of binary variables, perhaps recording whether or not a collection of objects arrayed in two-dimensional space have a specific dichotomous property such as a disease.
- An autologistic regression model, in which the log odds of the probability that one of the objects has the property, is regressed linearly on functions of the other variables.
An introduction to $p^*$

- *$P^*$ model is as follows* (Wasserman and Pattison, 1996)

\[
\Pr(X = x) = \frac{\exp\{\theta'z(x)\}}{\kappa(\theta)} = \frac{\exp\{\theta_1z_1(x) + \cdots + \theta_rz_r(x)\}}{\kappa(\theta)}
\]

$\theta_i$: a vector of the $r$ model parameters, the unknown ‘regression’ coefficients

$Z_r(x)$: the vector of the $r$ explanatory variables,

$K$: normalizing constant that ensures that the probabilities sum to unity.
Some parameters and graph statistics for $p^*$ models

<table>
<thead>
<tr>
<th>Type</th>
<th>Parameter</th>
<th>Graph statistic $z(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dyadic</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Choice</td>
<td>$\phi$</td>
<td>$L = \sum_{ij} X_{ij} = X_{++}$</td>
</tr>
<tr>
<td>Mutuality</td>
<td>$\rho$</td>
<td>$M = \sum_{i&lt;j} X_{ij}X_{ji}$</td>
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<tr>
<td><strong>Triadic</strong></td>
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<td></td>
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<tr>
<td>Transitivity</td>
<td>$\tau_T$</td>
<td>$T_T = \sum_{i,j,k} X_{ij}X_{jk}X_{ik}$</td>
</tr>
<tr>
<td>Intransitivity</td>
<td>$\tau_I$</td>
<td>$T_I = \sum_{i,j,k} X_{ij}X_{jk}(1 - X_{ik})$</td>
</tr>
<tr>
<td>Cyclicity</td>
<td>$\tau_C$</td>
<td>$T_C = \sum_{i,j,k} X_{ij}X_{jk}X_{ki}$</td>
</tr>
<tr>
<td>2-in-stars</td>
<td>$\sigma_I$</td>
<td>$S_I = \sum_{i,j,k} X_{ji}X_{ki}$</td>
</tr>
<tr>
<td>2-out-stars</td>
<td>$\sigma_O$</td>
<td>$S_O = \sum_{i,j,k} X_{ij}X_{ik}$</td>
</tr>
<tr>
<td>2-mixed-stars</td>
<td>$\sigma_M$</td>
<td>$S_M = \sum_{i,j,k} X_{ji}X_{ik}$</td>
</tr>
<tr>
<td><strong>Subgroup effects</strong></td>
<td>$\phi^{rs}$</td>
<td>$B^{rs} = \sum_{i,j} X_{ij} \delta_{ij,rs}$</td>
</tr>
<tr>
<td><strong>Individual level</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Differential expansiveness</td>
<td>$\alpha_i$</td>
<td>$X_{i+} = \text{outdegree (degree centrality)}$</td>
</tr>
<tr>
<td>Differential attractiveness</td>
<td>$\beta_i$</td>
<td>$X_{+i} = \text{indegree (degree prestige)}$</td>
</tr>
</tbody>
</table>

The indicator quantity $\delta_{ij,rs} = 1$ if $i$ is in the $r$th subgroup and $j$ is in the $s$th, and 0 otherwise.
Logit models

\[
\text{logit}(Y^*) = \log \left( \frac{\Pr(Y^* = 1)}{\Pr(Y^* = 0)} \right) = \beta_0 + \beta_1 Y_1 + \ldots + \beta_r Y_r
\]

\[
\frac{\Pr(Y^* = 1)}{\Pr(Y^* = 0)} = \exp \left( \beta_0 + \beta_1 Y_1 + \ldots + \beta_r Y_r \right) = e^{\beta_0} e^{\beta_1 Y_1} \ldots e^{\beta_r Y_r}
\]
The Logit $p^*$ representation: three sociomatrices

$X_{ij}^+$: the sociomatrix for relation $X$ where the tie from actor $i$ to actor $j$ is forced to be present.

$X_{ij}^-$: the sociomatrix for relation $X$ where the tie from actor $i$ to actor $j$ is forced to be absent.

$X_{ij}^C$: the sociomatrix of the complement relation for the tie from $i$ to $j$. This complement relation has no relational tie coded from $i$ to $j$—one can view this single tie as missing.
Logit $p^*$

$$\exp\{\omega_{ij}\} = \frac{\Pr(X_{ij} = 1|X_{ij}^c)}{\Pr(X_{ij} = 0|X_{ij}^c)}.$$  

Conditional odds (1)

$$\Pr(X_{ij} = 1|X_{ij}^c) = \exp\left\{\theta' z(x_{ij}^+)\right\} = \exp\left\{\theta' \left[ z(x_{ij}^+) - z(x_{ij}^-) \right] \right\}.$$  

$$\omega_{ij} = \log\left(\frac{\Pr(X_{ij} = 1|X_{ij}^c)}{\Pr(X_{ij} = 0|X_{ij}^c)}\right) = \theta' \left[ z(x_{ij}^+) - z(x_{ij}^-) \right].$$  

$$d_{ij}(z) = [z(x_{ij}^+) - z(x_{ij}^-)].$$  

$$\omega_{ij} = \theta' d_{ij}(z).$$
Parker and Asher (1993) Example

- Investigation about the relationship between elementary school children’s friendships quality and peer group (classroom) acceptance.
- 36 classrooms (12 third-grade, 10 fourth-grade, and 14 fifth-grade) =36 networks.
- Five public elementary schools in a midwestern US community.
- A total of 881 children.

- **Friendships quality networks:** three relations (very best friendship, best friendship, friendship) using self-choosing given a roster of the children.
- **Acceptance networks:** (e.g. how much classmates liked to play with each classmate) using a ‘roster-and-rating’.
- **Attribute data:** gender, age, race, loneliness using the self-report questionnaire.
- Study focusing on
  - mutual friendship relations (actor $I$ and actor $j$ both choose each other as friends).
  - study the differences between children with respect to acceptance and friendship quality.
Example: Single network

Table 5
Estimated parameters, (approximate) asymptotic standard errors, and pseudo-Wald statistics of the most complex model fit to the 552 dyads from the friendship data for the fourth-grade class.

<table>
<thead>
<tr>
<th>Effect</th>
<th>Explanatory variable</th>
<th>Model parameter</th>
<th>Estimated value</th>
<th>Approximate standard error</th>
<th>Wald_{PL}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choice</td>
<td>$L^{same}$</td>
<td>$\phi_{same}$</td>
<td>$-2.17$</td>
<td>1.15</td>
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<td>$L_{differ}$</td>
<td>$\phi_{differ}$</td>
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<td>Mutual</td>
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<td>$\rho^{gg}$</td>
<td>5.15</td>
<td>0.69</td>
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<td></td>
<td>$M^{bb}$</td>
<td>$\rho^{bb}$</td>
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<td>Attractiveness</td>
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**Table 7**

Estimated parameters, approximate standard errors, and pseudo-Wald statistics for parameters of the simplest model

<table>
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<tr>
<th>Effect</th>
<th>Explanatory variable</th>
<th>Parameter estimate</th>
<th>Standard error</th>
<th>Wald$_{PL}$</th>
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<tr>
<td>Attractiveness</td>
<td>$X_{+1}$</td>
<td>$-0.86$</td>
<td>$0.88$</td>
<td>$0.96$</td>
<td>$0.42$</td>
</tr>
<tr>
<td></td>
<td>$X_{+,23}$</td>
<td>$-0.36$</td>
<td>$0.91$</td>
<td>$0.15$</td>
<td>$0.70$</td>
</tr>
<tr>
<td></td>
<td>$X_{+,24}$</td>
<td>$0.00$</td>
<td>$0.00$</td>
<td>$-$</td>
<td>$1.00$</td>
</tr>
</tbody>
</table>
### Example: multi classes

Table 9

Model fit statistics for various models fit to all three classes simultaneously

<table>
<thead>
<tr>
<th>Model</th>
<th>All parameters, no restrictions (total)</th>
<th>Number parameters</th>
<th>$-2 \log(L)$</th>
<th>Change in fit</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3rd grade</td>
<td>(36)</td>
<td>315.59</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4th grade</td>
<td>(36)</td>
<td>336.02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5th grade</td>
<td>(36)</td>
<td>216.09</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Restrictions within classes (total)</td>
<td>25</td>
<td>958.28</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>3rd grade</td>
<td>(7)</td>
<td>348.23</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>4th grade</td>
<td>(6)</td>
<td>372.24</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>5th grade</td>
<td>(12)</td>
<td>237.81</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>$\rho_{3rd} = \rho_{4th} = \rho_{5th}$</td>
<td></td>
<td>960.33</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{3rd} = \gamma_{5th}$; $\sigma_{I,3rd} = \sigma_{I,4th}$</td>
<td></td>
<td></td>
<td>2.05</td>
</tr>
<tr>
<td></td>
<td>3a $\rho_{5th,gg} = 0$</td>
<td>20</td>
<td>968.88</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3b $[\gamma_{3rd} = \gamma_{5th}] = \gamma_{4th}$</td>
<td>20</td>
<td>970.68</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3c $[\tau_{T,3rd,gg} = \tau_{T,3rd,bb} = \tau_{T,3rd,gb}] = \tau_{T,4th}$</td>
<td>20</td>
<td>960.95</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3d $[\sigma_{I,3rd} = \sigma_{I,4th}] = \sigma_{I,5th}$</td>
<td>20</td>
<td>966.23</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3e $\sigma_{2,5th,gg} = 0$</td>
<td>20</td>
<td>963.27</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3f $\sigma_{M,4th} = \sigma_{M,5th,bb}$</td>
<td>20</td>
<td>960.33</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>3g $\sigma_{M,5th,bb} = [\sigma_{M,5th,gb} = \sigma_{M,5th,bg}]$</td>
<td>20</td>
<td>961.93</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>4 $[\tau_{T,3rd,gg} = \tau_{T,3rd,bb} = \tau_{T,3rd,gb}] = \tau_{T,4th}$</td>
<td>18</td>
<td>962.65</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>$[\sigma_{M,4th} = \sigma_{M,5th,bb}] = [\sigma_{M,5th,gb} = \sigma_{M,5th,bg}]$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>5a $\rho_{5th,gg} = 0$</td>
<td>17</td>
<td>970.84</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>5b $\sigma_{Q,5th,gg} = 0$</td>
<td>17</td>
<td>965.67</td>
<td>1</td>
</tr>
</tbody>
</table>

Change in fit for Model 4 is relative to Model 3f.
Table 10
Estimated parameters of models fit to multiple classes with restrictions on parameters within classes (Model 1 in Table 9)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter</th>
<th>Estimated value</th>
<th>Standard error</th>
<th>Wald_{PE}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choice</td>
<td>$\phi_{3rd}$</td>
<td>0.12</td>
<td>0.74</td>
<td>0.03</td>
</tr>
<tr>
<td></td>
<td>$\phi_{4th}$</td>
<td>2.65</td>
<td>0.93</td>
<td>8.14</td>
</tr>
<tr>
<td></td>
<td>$\phi_{5th}$</td>
<td>2.19</td>
<td>1.05</td>
<td>4.37</td>
</tr>
<tr>
<td>Mutuality</td>
<td>$\phi_{3rd}$</td>
<td>1.84</td>
<td>0.34</td>
<td>30.12</td>
</tr>
<tr>
<td></td>
<td>$\phi_{4th}$</td>
<td>1.84</td>
<td>0.29</td>
<td>40.10</td>
</tr>
<tr>
<td></td>
<td>$\phi_{5th}$</td>
<td>1.48</td>
<td>0.46</td>
<td>10.33</td>
</tr>
<tr>
<td></td>
<td>$\phi_{5th,bb}$</td>
<td>4.82</td>
<td>1.15</td>
<td>17.45</td>
</tr>
<tr>
<td>Acceptance ratings</td>
<td>$\gamma_{3rd}$</td>
<td>1.12</td>
<td>0.22</td>
<td>25.73</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{4th}$</td>
<td>0.67</td>
<td>0.21</td>
<td>10.34</td>
</tr>
<tr>
<td></td>
<td>$\gamma_{5th}$</td>
<td>1.54</td>
<td>0.35</td>
<td>19.04</td>
</tr>
<tr>
<td>Degree centralization</td>
<td>$\alpha_{5th}$</td>
<td>4.92</td>
<td>1.89</td>
<td>6.77</td>
</tr>
<tr>
<td>Transitivity</td>
<td>$\tau_{3rd,bb} = \tau_{3rd,gb} = \tau_{3rd,bb}$</td>
<td>0.25</td>
<td>0.03</td>
<td>67.53</td>
</tr>
<tr>
<td></td>
<td>$\tau_{4th}$</td>
<td>0.31</td>
<td>0.04</td>
<td>64.75</td>
</tr>
<tr>
<td></td>
<td>$\tau_{5th}$</td>
<td>0.50</td>
<td>0.10</td>
<td>27.35</td>
</tr>
<tr>
<td>2-in-stars</td>
<td>$\sigma_{3rd}$</td>
<td>-0.21</td>
<td>0.07</td>
<td>9.43</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{4th}$</td>
<td>-0.31</td>
<td>0.09</td>
<td>11.67</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{5th}$</td>
<td>-0.61</td>
<td>0.16</td>
<td>14.62</td>
</tr>
<tr>
<td>2-out-stars</td>
<td>$\sigma_{O,3rd}$</td>
<td>0.10</td>
<td>0.04</td>
<td>5.45</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{O,5th,bb} = \sigma_{O,5th,gb} = \sigma_{O,5th,bb}$</td>
<td>-0.31</td>
<td>0.12</td>
<td>6.78</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{O,5th,gg}$</td>
<td>0.40</td>
<td>0.24</td>
<td>2.71</td>
</tr>
<tr>
<td>2-mixed-stars</td>
<td>$\sigma_{M,3rd}$</td>
<td>-0.20</td>
<td>0.03</td>
<td>32.25</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{M,4th}$</td>
<td>-0.36</td>
<td>0.05</td>
<td>48.99</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{M,5th,bb}$</td>
<td>-0.32</td>
<td>0.08</td>
<td>17.23</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{M,5th,gb} = \sigma_{M,5th,bb}$</td>
<td>-0.45</td>
<td>0.10</td>
<td>20.14</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{M,5th,gg}$</td>
<td>-1.08</td>
<td>0.23</td>
<td>21.34</td>
</tr>
</tbody>
</table>
Example: multi classes

Table 11
Estimated parameters of multiple class model (Model 4 in Table 9)

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter</th>
<th>Estimated value</th>
<th>Standard error</th>
<th>Odds ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Choice</td>
<td>$\phi_{3rd}$</td>
<td>0.54</td>
<td>0.68</td>
<td>1.71</td>
</tr>
<tr>
<td></td>
<td>$\phi_{4th}$</td>
<td>2.56</td>
<td>0.58</td>
<td>12.96</td>
</tr>
<tr>
<td></td>
<td>$\phi_{5th}$</td>
<td>1.44</td>
<td>0.74</td>
<td>4.22</td>
</tr>
<tr>
<td>Mutuality</td>
<td>$\rho_{3rd} = \rho_{4th} = \rho_{5th}$</td>
<td>1.81</td>
<td>0.20</td>
<td>6.12</td>
</tr>
<tr>
<td></td>
<td>$\rho_{5th,gg}$</td>
<td>2.74</td>
<td>1.10</td>
<td>15.55</td>
</tr>
<tr>
<td>Degree Centralization</td>
<td>$\alpha_{5th}$</td>
<td>4.37</td>
<td>1.78</td>
<td>79.39</td>
</tr>
<tr>
<td>Acceptance</td>
<td>$\gamma_{3rd} = \gamma_{5th}$</td>
<td>1.32</td>
<td>0.17</td>
<td>3.73</td>
</tr>
<tr>
<td>Ratings</td>
<td>$\gamma_{4th}$</td>
<td>0.62</td>
<td>0.17</td>
<td>1.87</td>
</tr>
<tr>
<td>Transitivity</td>
<td>$\tau_{T,3rd,gg} = \tau_{T,3rd,bb} = \tau_{T,3rd,gb} = \tau_{T,4th}$</td>
<td>0.28</td>
<td>0.02</td>
<td>1.33</td>
</tr>
<tr>
<td></td>
<td>$\tau_{T,5th}$</td>
<td>0.55</td>
<td>0.06</td>
<td>1.73</td>
</tr>
<tr>
<td>In-2-Stars</td>
<td>$\sigma_{I,3rd} = \sigma_{I,4th}$</td>
<td>$-0.27$</td>
<td>0.05</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{I,5th}$</td>
<td>$-0.53$</td>
<td>0.11</td>
<td>0.59</td>
</tr>
<tr>
<td>Out-2-Stars</td>
<td>$\sigma_{O,3rd}$</td>
<td>0.09</td>
<td>0.04</td>
<td>1.10</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{O,5th,bb} = \sigma_{O,5th,gb} = \sigma_{O,5th,bg}$</td>
<td>$-0.27$</td>
<td>0.10</td>
<td>0.76</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{O,5th,gg}$</td>
<td>0.38</td>
<td>0.24</td>
<td>1.47</td>
</tr>
<tr>
<td>Mixed-2-Stars</td>
<td>$\sigma_{M,3rd}$</td>
<td>$-0.20$</td>
<td>0.03</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{M,4th} = \sigma_{M,5th,bb} = \sigma_{M,5th,gb} = \sigma_{M,5th,bg}$</td>
<td>$-0.35$</td>
<td>0.04</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{M,5th,gg}$</td>
<td>$-0.99$</td>
<td>0.21</td>
<td>0.37</td>
</tr>
</tbody>
</table>
Single and Multi network analysis:
Parker and Asher (1993) Example

- Multi network P* model
  - Finding communalities and similarities among networks under study
  - Also, finding uniqueness and idiosyncrasies within network
Questions